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RESEARCH NOTES

A NOTE ON THE UNIQUE SOLVABILITY OF A CLASS OF NONLINEAR EQUATIONS

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ABSTRACT. The aim of the present note is to devise a simple criterion for the existence of the unique solution of a class of nonlinear equations whose solvability is taken for granted.

KEY WORDS AND PHRASES. Hammerstein equation, Monotonically Decomposable operators.
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1. INTRODUCTION.

In equations describing physical problems presence of more than one solution may sometimes create complications. One is often led to a solution that may differ from the desired solution and hence a lack of agreement of the solution with the experimental result occurs. We are therefore motivated in devising a simple criterion by which the uniqueness of the solution of a class of equations is guaranteed.

In what follows we take X to be a complete supermatic space [1] and f to be an element of X.

A is a nonlinear mapping of X into X and we are interested in solving the equation

$$u = Au + f, \quad f \in X \tag{1.1}$$

by the iterates of the form

$$u_{n+1} = A u_n + f \quad (u_0 \text{ prechosen}) \tag{1.2}$$

Section 2 contains the convergence theorem and an example is appended in section 3. Earlier Sen [2], Sen and Mukherjee [3] proved the unique solvability of the nonlinear equation $Au = Pu$ in the setting of a metric space.

2. CONVERGENCE.

THEOREM 2.1. Let the following conditions be fulfilled:

i) There exists a bounded linear operator L mapping X into X s.t.

a) $\rho\ (Au, Av) \leqq \rho(Lu, Lv),\ \forall\ u, v \in X$

b) $\rho\ (L^p Au, L^p Av) \leqq \rho(L^{p+1}u, L^{p+1}v),\ p = 0,1, \ldots(m-1)$

c) $\rho\ (L^m u, L^m v) \leqq q\ \rho(u,v),\ \forall\ u, v \in X,\ 0 < q < 1$ and for fixed m

ii) f belongs to the range of $(I-A)$.

Then the sequence u_n defined by (1.2) converges to the unique solution of (1.1).

PROOF. By condition (ii) there exists a

$$u^* \in X \quad s.t \quad u^* = Au^* + f \tag{2.1}$$

The space being supermetric and the use of the conditions a), b) and c) yields

$$\begin{aligned} \rho(u_{m+1}, u^*) &= \rho\ (Au_m, Au^*) \\ &\leqq \rho(L^{(m-1)}u_o, L^{(m-1)}u^*) \\ &\leqq q\ \rho(Lu_o, Lu^*) \end{aligned} \tag{2.2}$$

Hence $\rho(u_{nm}, u^*) \leqq q^n \rho(u_o, u^*)$ --- 0 as $n \to \infty$ $(0 < q < 1)$ (2.3)

If v^* is another solution of the equation

$$\begin{aligned} \rho\ (u^*, v^*) &= \rho\ (Au^* + f, Av^* + f) \\ &\leqq q\ \rho(u^*, v^*)\ (0 < q < 1) \end{aligned} \tag{2.4}$$

Hence, $u^* = v^*$

Therefore $\{u_n\}$ converges uniquely to the solution of (1.1) where f belongs to the range of (I.A).

3. EXAMPLE.

In this section we consider the following Hammerstein equation

$$u(x) = 1 + \int_0^1 |x-t|\ [\ u(t) - \frac{1}{2} u^2(t)\ ]\ dt \tag{3.1}$$

in the setting of C(0,1).

By using the theory of Monotonically Decomposable Operator (MDO) [1] Collatz proved [4] the existence of a solution $u(x)$ with

$$2(x - x^2) \leqq u(x) \leqq 2(1 - x - x^2) \tag{3.2}$$

Let $X = \{u(x)\ /\ 2(x - x^2) \leqq u(x) \leqq 2(1 - x - x^2)\}$

We would show that in X the equation (3.1) admits of a unique solution. Here

$$Au = \int_0^1 |x-t|\ [\ u(t) - \frac{1}{2} u^2(t)\ ]\ dt \tag{3.3}$$

Let us choose $Lu = \int_0^1 |x-t|\ u(t)\ dt$ (3.4)

We take $\rho\ (u,v) = \|\ u-v\ \| = \max\limits_{0 \leq x \leq 1} |u(x) - v(x)| \quad 0 \leqq x \leqq 1$ (3.5)

$\forall\ u(x), v\ (x)\quad C(0,1).$

Let us consider the metric in X induced by the metric in C(0,1) and Complete X w.r.t. the induced metric so that X is a complete supermetric space.

$$Au - Av = \int_0^1 |x-t| \ (u(t)-v(t))dt$$
$$- \frac{1}{2} (u(\xi) + v(\xi)) \int_0^1 |x-t| \ u(t) \ dt$$
$$- \frac{1}{2} (u(\eta) + v(\eta)) \int_0^1 |x-t| \ v(t) \ dt \quad (3.6)$$
$$0 < \xi < 1$$
$$0 < \eta < 1$$

$$\max_{0 \leq x \leq 1} \left| 1 - \frac{u(x) + v(x)}{2} \right| \leq 1, \quad \forall \ u(x), v(x) \in X \quad (3.7)$$

$$\rho \ (Lu,Lv) = \max_{0 \leq x \leq 1} |u(\bar{\xi})-v(\bar{\eta})| \int_0^1 |x-t| \ dt \ (0<\bar{\xi}<1, \ o<\bar{\eta}<1) \quad (3.8)$$

$$\int_0^1 |x-t| \ v(t) \ dt \leq \frac{v(\bar{\eta})}{|u(\bar{\xi}) - v(\bar{\eta})|} \ \rho(Lu,Lv) \quad (3.9)$$

Neglecting quantities of second order in ξ,η

$$|Au-Av| \leq [\ 1 - \frac{1}{2} \ (u(\xi) + v(\xi) \ ) + v(\bar{\eta})] \ \rho(Lu, \ Lv)$$

$$\text{Now} \quad 1 - \frac{1}{2} \ (u(\xi) + v(\xi) + v(\bar{\eta}))$$

$$1 - 2 \ (\xi - \xi^2) + 2 \ (1 - \bar{\eta} + \bar{\eta}^2) \simeq 1 \quad (3.11)$$

Therefore $\rho(Au, Av) \leq \rho(Lu, Lv), \ \forall u, v \in X$ (3.12)

Simple manipulation shows that

$$\rho(LAu, \ LAv) \leq \rho(L^2u, \ L^2v) \quad (3.13)$$

Using Schwartz inequality we have

$$\rho(L^2u, \ L^2v) \leq \frac{1}{3} \ \rho(u,v) \quad (3.14)$$

It may however be noted that both A and L map X into X where X is a complete metric space.

Hence by Theorem 2.1 $\{u_n\}$ defined by

$$u_{n+1} = 1 - \int_0^1 |x-t| \ [u_n(t) - \frac{1}{2} u_n^2(t))] \ dt \quad n = 0,1,2,...$$

Converges to the unique solution of equation (3.1) in X.

3.1. In many nonlinear equations it is possible to generate L's which could be prototypes of the linearized versions of A.

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