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NEW CLASSIFICATION OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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ABSTRACT. New classification of analytic functions with negative coefficients is given by using the coefficients inequality, that is, new subclass $A(p,n,B_k)$ of analytic functions with negative coefficient is defined. The object of the present paper is to prove various distortion theorems for functions in $A(p,n,B_k)$, and for fractional calculus of functions belonging to $A(p,n,B_k)$. Further, some properties of the class $A(p,n,B_k)$ are shown.

KEYWORDS AND PHRASES. Analytic function, distortion theorem, fractional integral, fractional derivative, extreme point.
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I. INTRODUCTION.

Let $A_{p,n}$ denote the class of functions of the form

$$f(z) = z^p - \sum_{k=p+n}^{\infty} a_k z^k \qquad (a_k \geq 0;\ p \in N;\ n \in N) \qquad (1.1)$$

which are analytic in the unit disk $U = \{z: |z| < 1\}$, where $N = \{1,2,3,\ldots.\}$.
A function $f(z)$ belonging to $A_{p,n}$ is said to be in the class $S_{p,n}(\alpha)$ if and only if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha \qquad (1.2)$$

for some α $(0 \leq \alpha < p)$, and for all $z \in U$. Also, function $f(z)$ belonging to $A_{p,n}$ is said to be in the class $K_{p,n}(\alpha)$ if and only if

$$\text{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \tag{1.3}$$

for some α $(0 \leq \alpha < p)$, and for all $z \in U$.

We note that $S_{p,n}(\alpha)$ and $K_{p,n}(\alpha)$ are the subclasses of p-valent starlike functions and p-valent convx functions of order α, respectively. Furthermore, we note that $S_{p,n}(\alpha) \subseteq S_{p,n}(0)$, $K_{p,n}(\alpha) \subseteq K_{p,n}(0)$ for $0 \leq \alpha > p$, and that $f(z) \in K_{p,n}(\alpha)$ if and only if $zf'(z)/p \in S_{p,n}(\alpha)$ for $0 \leq \alpha < p$.

In view of the results by Owa [1], we know that $f(z) \in S_{p,n}(\alpha)$ if and only if

$$\sum_{k=p+n}^{\infty} (k - \alpha)\, a_k \leq p - \alpha,$$

and that $f(z) \in K_{p,n}(\alpha)$ if and only if

$$\sum_{k=p+n}^{\infty} k(k - \alpha)\, a_k \leq p - \alpha.$$

Let $A(p,n,B_k)$ denote the subclass of $A_{p,n}$ consisting of functions which satisfy the following inequality

$$\sum_{k=p+n}^{\infty} B_k a_k \leq 1 \qquad (B_k > 0). \tag{1.4}$$

It follows from (1.4) that

$$A(p,n,B_k) \subseteq A(p,n,C_k) \qquad (0 < C_k \leq B_k).$$

Therefore we can classify the analytic functions belonging to $A_{p,n}$ according to the above inequality (1.4).

REMARK 1. $A(1,1,(k-\alpha)/(1-\alpha)) = T^*(\alpha)$ (Silverman [2]), $A(1,1,k(k-\alpha)/(1-\alpha)) = C(\alpha)$ (Silverman [2]), $A(1,1,k(1+\beta)/2\beta(1-\alpha)) = P^*(\alpha,\beta)$ (Gupta and Jain [3]), $A(1,1\ \{(k-1)+\beta(k+1-2\alpha)\}/2\beta(1-\alpha)) = S^*(\alpha,\beta)$ (Gupta and Jain [4]), $A(1,1,k\{k-1)+\beta(k+1-2\alpha)\}/2\beta(1-\alpha)) = C^*(\alpha,\beta)$ (Gupta and Jain [4]), $A(1,1,1/(1-\alpha)) = R(\alpha)$ (Sarangi and Uralegaddi [5]), $A(1,1,k/(1-\alpha)) = Q(\alpha)$ (Sarangi and Uralegaddi [5]), $A(1,1,(k+m-1)!(2k+m-1)/(k-1)!(m+1)!) = K_m^*$ (Owa [6]), $A(1,1,(m+\alpha k+k)\Gamma(m+\alpha+k)/(k-1)!\Gamma(m+\alpha+2)) = R(m+\alpha)$ (Owa [7]), $A(1,1,(k-\beta)C(\alpha,k)/(1-\beta)) = R[\alpha,\beta]$ (Silverman and Silvia [8]), $A(1,1,k(1+\gamma)C(\alpha,k)/2\gamma(1-\beta)) = P_\alpha[\beta,\gamma]$ (Owa and Ahuja [9]), and $A(1,1,(m+\alpha+2k-1)\Gamma(m+\alpha+k)/(k-1)!\Gamma(m+2+\alpha)) = K(m+\alpha)$ (Owa [10]), where $C(\alpha,k) = \prod_{j=2}^{k} (j-2\alpha)/(k-1)!$.

REMARK 2. $A(1,n,(k-\alpha)/(1-\alpha)) = {}_\alpha(n)$ (Chatterjea [11]), $A(1,n,k(k-\alpha)/(1-\alpha)) = C_\alpha(n)$ (Chatterjea [11]), and $A(1,n,k/(1-\alpha)) = C(\alpha,n)$ (Sekine and Owa [12]).

REMARK 3. $A(p,1,(1+b)k/2b(1-a)p) = {}_p(a,b)$ (Owa [13]), $A(p,1,(1+b)k^2/2b(1-a)p^2) = C_p(a,b)$ (Owa [13]), $A(p,1,(1-ap+bk)/(b-a)p) = T_p^*(a,b)$ (Goel and Sohi [14]), $A(p,1,k(1-ap+bk)/(b-a)p^2) = C_p(a,b)$ (Goel and Sohi [14]),

$A(p,1,(k+m-1)!(2k+m-p)/(k-p)!(m+p)!) = K^{*}_{m+p-1}$ (Owa [15]),
$A(p,1,k(1+\beta)/(\beta-\alpha)p) = T^{*}(p,\alpha,\beta)$(Shukla and Dashrath [16]),
$A(p,1,k^2(1+\beta)/(\beta-\alpha)p^2) = C(p,\alpha,\beta)$ (Owa and Srivastava [17]),
$A(p,1,(k-\alpha)/(p-\alpha)) = T^{*}(p,\alpha)$ (Owa [1]), and $A(p,1,k(k-\alpha)/p(p-\alpha)) = C(p,\alpha)$ (Owa [1]).

2. DISTORTION THEOREMS.

We begin with the statement and the proof of the following result.

THEOREM 1. Let the fnction $f(z)$ defined by (1.1) be in the class $A(p,n,B_k)$ with $B_k \leq B_{k+1}$. Then

$$\max\{0, |z|^p - \frac{1}{B_{p+n}}|z|^{p+n}\} \leq |f(z)| \leq |z|^p + \frac{1}{B_{p+n}}|z|^{p+n} \tag{2.1}$$

for $z \in U$. The equalities in (2.1) are attained for the function $f(z)$ given by

$$f(z) = z^p - \frac{1}{B_{p+n}} z^{p+n}. \tag{2.2}$$

PROOF. Since $f(z) \in A(p,n,B_k)$ and $B_k \leq B_{k+1}$, we have

$$B_{p+n}\sum_{K=p+n}^{\infty} a_k \leq \sum_{k=p+n}^{\infty} B_k a_k \leq 1, \tag{2.3}$$

or

$$\sum_{k=p+n}^{\infty} a_k \leq \frac{1}{B_{p+n}}. \tag{2.4}$$

Hence, it follows from (2.4) that

$$|f(z)| \geq \max\{0, |z|^p - |z|^{p+n}\sum_{k=p+n}^{\infty} a_k\}$$

$$\geq \max\{0, |z|^p - \frac{1}{B_{p+n}}|z|^{p+n}\} \tag{2.5}$$

and

$$|f(z)| \leq |z|^p + |z|^{p+n}\sum_{k=p+n}^{\infty} a_k$$

$$\leq |z|^p + \frac{1}{B_{p+n}}|z|^{p+n}. \tag{2.6}$$

Furthermore, it is clear that the equalities in (2.1) are attained the function $f(z)$ given by (2.2).

REMARK 4. Note that if $B_{p+n} \geq 1$, then

$$\max\{0, |z|^p - \frac{1}{B_{p+n}}|z|^{p+n}\} = |z|^p - \frac{1}{B_{p+n}}|z|^{p+n} \quad (z \in U).$$

From [1], $f(z)$ is p-valent starlike in U if and only if $B_{p+n} \geq (p+n)/p$. Therefore, we have

$$|z|^p - \frac{1}{B_{p+n}}|z|^{p+n} \leq |f(z)| \leq |z|^p + \frac{1}{B_{p+n}}|z|^{p+n}$$

for p-valent starlike functions of the form (1.1).

THEOREM 2. Let the function $f(z)$ defined by (1.1) be in the class $A(p,n,kB_k)$ with $B_k \leq B_{K+1}$. Then

$$\max\{0,\ p|z|^{p-1} - \frac{1}{B_{p+n}}|z|^{p+n-1}\} \leq |f'(z)| \leq p|z|^{p-1} + \frac{1}{B_{p+n}}|z|^{p+n-1} \tag{2.7}$$

for $z \in U$. The equalities in (2.7) are attained for the function $f(z)$ given by

$$f(z) = z^p - \frac{1}{(p+n)B_{p+n}} z^{p+n} \tag{2.8}$$

PROOF. Note that, for $f(z) \in A(p,n,kB_k)$ and $B_k \leq B_{k+1}$,

$$B_{p+n}\sum_{k=p+n}^{\infty} ka_k \leq \sum_{k=p+n}^{\infty} kB_k a_k \leq 1, \tag{2.9}$$

that is, that

$$\sum_{k=p+n}^{\infty} ka_k \leq \frac{1}{B_{p+n}}. \tag{2.10}$$

This gives that

$$|f'(z)| \geq \max\{0,\ p|z|^{p-1} - |z|^{p+n-1}\sum_{k=p+n}^{\infty} ka_k\}$$

$$\geq \max\{0,\ p|z|^{p-1} - \frac{1}{B_{p+n}}|z|^{p+n-1}\} \tag{2.11}$$

and

$$|f'(z)| \leq p|z|^{p-1} + |z|^{p+n-1}\sum_{k=p+n}^{\infty} ka_k$$

$$\leq p|z|^{p-1} + \frac{1}{B_{p+n}}|z|^{p+n-1}. \tag{2.12}$$

Further, the equalities in (2.7) are attained for the function $f(z)$ given by (2.8).

REMARK 5. If $B_{p+n} \geq 1/p$, then

$$\max\{0,\ p|z|^{p-1} - \frac{1}{B_{p+n}}|z|^{p+n-1}\} = p|z|^{p-1} - \frac{1}{B_{p+n}}|z|^{p+n-1}$$

for $z \in$. Thus, from [17], we know that, for p-valent starlike functions of the form (1.1), Theorem 2 gives

$$p|z|^{p-1} - \frac{1}{B_{p+n}}|z|^{p+n-1} \leq |f'(z)| \leq p|z|^{p-1} + \frac{1}{B_{p+n}}|z|^{p+n-1}.$$

Next, we derive the following lemma.

LEMMA 1. Let

$$\prod_{i=1}^{j}(k-1+i) = \sum_{i=1}^{j} A_i k^i \tag{2.13}$$

for $j \geq 2$. . Then we have

$$\prod_{i=1}^{j} A_i (p+n)^{i-} = \prod_{i=2}^{j}(p+n-1+i). \tag{2.14}$$

PROOF. In case of $j=2$, it is clear from (2.13) that

$$\prod_{i=1}^{2} (k - 1 + i) = k(k + 1) = k^2 + k, \tag{2.15}$$

or, that $A_1 = 1$ and $A_2 = 1$. Thus we have

$$\sum_{i=1}^{2} A_i (p + n)^{i-1} = 1 + (p + n) = p + n + 1 \tag{2.16}$$

which proves (2.14) for j=2.

Assume that (2.14) holds true for j=j. Then

$$\begin{aligned}\prod_{i=1}^{j+1} (k - 1 + i) &= (k + j) \prod_{i=1}^{j} (k - 1 + i)\\ &= (k + j) \left(\sum_{i=1}^{j} A_i k^i \right) = \sum_{i=1}^{j+1} B_i k^i ,\end{aligned} \tag{2.17}$$

where

$$B_1 = jA_1,\ B_{j+1} = A_j,\ B_i = A_{i-1} + jA_i \ (i = 2,3,\ldots,j). \tag{2.18}$$

Hence we obtain

$$\begin{aligned}\sum_{i=1}^{j+1} B_i (p + n)^{i-1} &= jA_1 + \sum_{i=2}^{j} (A_{i-1} + jA_i)\ (p+n)^{i-1} + A_j (p+n)^j\\ &= j \sum_{i=1}^{j} A_i (p+n)^{i-1} + (p+n) \sum_{i=1}^{j} A_i (p+n)^{i-1}\\ &= (p + n + j) \sum_{i=1}^{j} A_i (p + n)^{i-1}\\ &= (p + n + j) \prod_{i=2}^{j} (p + n - 1 + i)\\ &= \prod_{i=2}^{j+1} (p + n - 1 + i).\end{aligned} \tag{2.19}$$

Consequently, by the mathematical induction, we complete the proof of Lemma 1.

Applying Lemma 1, we prove

THEOREM 3. Let the function f(z) defined by (1.1) be in the class $A(p,n,k^m B_k)$ with $B_k \leq B_{k+1}$ and $2 \leq m \leq p$. Then we have

$$|f^{(j)}(z)| \geq \max \left\{0, \left(\prod_{i=1}^{j} (p+1-i)\right) |z|^{p-j} - \left(\frac{\prod_{i=2}^{j} (p+n-1+i)}{(p+n)^{m-1} B_{p+n}}\right) |z|^{p+n-j}\right\} \tag{2.20}$$

and

$$|f^{(j)}(z)| \geq \left(\prod_{i=1}^{j} (p+1-i)\right) |z|^{p-j} + \left(\frac{\prod_{i=2}^{j} (p+n-1+i)}{(p+n)^{m-1} B_{p+n}}\right) |z|^{p+n-j} \tag{2.21}$$

for z e and $2 \leq j \leq m$.

PROOF. Since $f(z)$ e $A(p,n,k^m B_k)$ and $B_k \leq B_{k+1}$, we note that

$$(p + n)^{m-t} B_{p+n} \sum_{k=p+n}^{\infty} k^t a_k \geq \sum_{k=p+n}^{\infty} k^m B_k a_k \leq 1, \tag{2.22}$$

that is, that

$$\sum_{k=p+n}^{\infty} k^t a_k \leq \frac{1}{(p+n)^{m-t} B_{p+n}} \tag{2.23}$$

for $2 \leq t \leq m$. For $f(z)$ defined by (1.1), we have

$$f^{(j)}(z) = \left(\prod_{i=1}^{j}(p+1-i)\right)z^{p-j} - \sum_{k=p+n}^{\infty}\prod_{i=1}^{j}(k+1-i))a_k z^{k-j} \tag{2.24}$$

for $2 \leq j \leq m \leq p$. Hence, by using Lemma 1 and (2.23), we obtain

$$|f^{(j)}(z)| \leq \left(\prod_{i=1}^{j}(p+1-i)\right)|z|^{p-j} + |z|^{p+n-j}\sum_{k=p+n}^{\infty}\left(\prod_{i=1}^{j}(k-1+i)\right)a_k$$

$$= \left(\prod_{i=1}^{j}(p+1-i)\right)|z|^{p-j} + |z|^{p+n-j}\sum_{k=p+n}^{\infty}\left(\sum_{i=1}^{j}A_i k^i\right)a_k$$

$$= \left(\prod_{i=1}^{j}(p+1-i)\right)|z|^{p-j} + |z|^{p+n-j}\sum_{i=1}^{j}A_i\left(\sum_{k=p+n}^{\infty}k^i a_k\right)$$

$$\leq \prod_{i=1}^{j}(p+1-i)\,|z|^{p-j} + |z|^{p+n-j}\sum_{i=1}^{j}\left(\frac{A_i}{(p+n)^{m-i}B_{p+n}}\right)$$

$$= \left(\prod_{i=1}^{j}(p+1-i)\right)|z|^{p-j} + \frac{|z|^{p+n-1}}{(p+n)^{m-1}B_{p+n}}\left(\sum_{i=1}^{j}A_i(p+n)^{i=1}\right)$$

$$= \left(\prod_{i=1}^{j}(p+1-i)\right)|z|^{p-j} + \frac{\prod_{i=2}^{j}(p+n-1+i)}{(p+n)^{m-1}B_{p+n}}|z|^{p=n-j} \tag{2.25}$$

which shows (2.21). Similarly,

$$|f^{(j)}(z)| \geq \max\left\{0,\left(\prod_{i=1}^{j}(p+1-i)\right)|z|^{p-j} - |z|^{p=n-j}\sum_{k=p+n}^{\infty}\left(\prod_{i=1}^{j}(k-1+i)\right)a_k\right\}$$

$$= \max\left\{0,\left(\prod_{i=1}^{j}(p+1-i)\right)|z|^{p-j} - \frac{\prod_{i=2}^{j}(p+n-1+i)}{(p+n)^{m-1}B_{p+n}}|z|^{p+n-j}\right\} \tag{2.26}$$

which gives (2.20). Thus we have the theorem.

REMARK 6. If $B_{p+n} \geq \left(\prod_{i=2}^{j}(p+n-1+i)\right)/(p+n)^{m-1}\prod_{i=1}^{j}(p+1-i)$, then

$$\max\left\{0,\left(\prod_{i=1}^{j}(p+1-i)\right)|z|^{p-j} - \frac{\prod_{i=2}^{j}(p+n-1+i)}{(p+n)^{m-1}B_{p+n}}|z|^{p+n-j}\right\}$$

$$= \left(\prod_{i=1}^{j}(p+1-i)\right)|z|^{p-j} - \frac{\prod_{i=2}^{j}(p+n-1+i)}{(p+n)^{m-1}B_{p+n}}|z|^{p+n-j}\,.$$

THEOREM 4. Let the function $f(z)$ defined by (1.1) be in the class $A(p,n,k^m B_k)$ with $B_k \leq B_{k+1}$ and $p+1 \leq m \leq p+n$. Then we have

$$\left|f^{(j)}(z)\right| \leq \frac{\prod_{i=2}^{j}(p+n-1+i)}{(p+n)^{m-1}B_{p+n}}\left|z\right|^{p+n-j} \tag{2.27}$$

for $z \in U$ and $p+1 \leq j \leq m$.

PROOF. Note that

$$\sum_{k=p+n}^{\infty} k^{t} a_k \leq \frac{1}{(p+n)^{m-t}B_{p+n}} \tag{2.28}$$

for $p+1 \leq t \leq m$. Since

$$f^{(j)}(z) = -\sum_{k=p+n}^{\infty}\left(\prod_{i=1}^{j}(k+1-i)\right)a_k z^{k-j} \tag{2.29}$$

for $p+1 \leq j \leq m$, by using Lemma 1 and (2.28), we have

$$\left|f^{(j)}(z)\right| \leq \left|z\right|^{p+n-j}\sum_{k=p+n}^{\infty}\left(\prod_{i=1}^{j}(k-1+i)\right)a_k$$

$$\leq \frac{\prod_{i=2}^{j}(p+n-1+i)}{(p+n)^{m-1}B_{p+n}}\left|z\right|^{p+n-j}, \tag{2.30}$$

which completes the proof of Theorem 4.

3. FRACTIONAL CALCULUS.

Many essentially equivalent definitions of the fractional calculus, that is, the fractional derivatives and the fractional integrals, have been in the literature (cf., [18], [19], [20], and [21]). We find it to be convenient to recall here the following definitions which were used recently by Owa ([22], [23]).

DEFINITION 1. The fractional integral or order λ is defined by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)}\int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}}\,d\zeta, \tag{3.1}$$

where $\lambda > 0$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)$ & 0.

DEFINITION 2. The fractional derivative of order λ is defined by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}}\,d\zeta, \tag{3.2}$$

where $0 \leq \lambda < 1$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

DEFINITION 3. Under the hypotheses of Definition 2, the fractional derivative of order $(n+\lambda)$ is defined by

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z), \tag{3.3}$$

where $0 \leq \lambda < 1$ and $n \in N_0 = N \cup \{0\}$.

With the above definitions of the fractional calculus, we prove

THEOREM 5. Let the funcion $f(z)$ defined by (1.1) be in the class $A(p,n,B_k)$ with $B_k \leq B_{k+1}$. Then

$$|D_z^{-\lambda} f(z)| \geq \max\{0, \frac{\Gamma(p+1)|z|^{p+\lambda}}{\Gamma(p+1+\lambda)}(1 - \frac{\Gamma(p+n+1)\ \Gamma(p+1+\lambda)}{\Gamma(p+n+1+\lambda)\ \Gamma(p+1) B_{p+n}} |z|^n)\} \tag{3.4}$$

and

$$|D_z^{-\lambda} f(z)| \leq \frac{\Gamma(p+1)\ |z|^{p+\lambda}}{\Gamma(p+1+\lambda)}(1 + \frac{\Gamma(p+n+1)\ \Gamma(p+1+\lambda)}{\Gamma(p+n+1+\lambda)\ \Gamma(p+1) B_{p+n}} |z|^n) \tag{3.5}$$

for $\lambda > 0$ and $z \in U$. The equalities in (3.4) and (3.5) are attained for the function $f(z)$ given by (2.2).

PROOF. We define the function $F(z)$ by

$$\begin{aligned} F(z) &= \frac{\Gamma(p+1+\lambda)}{\Gamma(p+1)} z^{-\lambda} D_z^{-\lambda} f(z) \\ &= z^p - \sum_{k=p+n}^{\infty} \frac{\Gamma(k+1)\ \Gamma(p+1+\lambda)}{\Gamma(k+1+\lambda)\ \Gamma(p+1)} a_k z^k \end{aligned} \tag{3.6}$$

for $\lambda > 0$. Then the function $\phi(k)$ defined by

$$\phi(k) = \frac{\Gamma(k+1)\ \Gamma(p+1+\lambda)}{\Gamma(k+1+\lambda)\ \Gamma(p+1)} \qquad (k \geq p+n) \tag{3.7}$$

is decreasing in k. Hence we have

$$0 < \phi(k) \leq \phi(p+n) = \frac{\Gamma(p+n+1)\ \Gamma(p+1+\lambda)}{\Gamma(p+n+1+\lambda)\ \Gamma(p+1)} . \tag{3.8}$$

Therefore, it follows from (2.4) and (3.8) that

$$\begin{aligned} |F(z)| &\geq \max\{0, |z|^p - \phi(p+n)|z|^{p+n} \sum_{k=p+n}^{\infty} a_k\} \\ &\geq \max\{0, |z|^p - \frac{\Gamma(p+n+1)\ \Gamma(p+1+\lambda)}{\Gamma(p+n+1+\lambda)\ \Gamma(p+1)\ B_{p+n}} |z|^{p+n}\} \end{aligned} \tag{3.9}$$

which implies (3.4), and

$$\begin{aligned} |F(z)| &\leq |z|^p + \phi(p+n)\ |z|^{p+n} \sum_{k=p+n}^{\infty} a_k \\ &\leq |z|^p + \frac{\Gamma(p+n+1)\ \Gamma(p+1+\lambda)}{\Gamma(p+n+1+\lambda)\ \Gamma(p+1)\ B_{p+n}} |z|^{p+n} \end{aligned} \tag{3.10}$$

which gives (3.5).

Furthermore, since the equalities in (3.9) and (3.10) are attained the function f(z) defined by

$$D_z^{-\lambda} f(z) = \frac{\Gamma(p+1)\ z^{p+\lambda}}{\Gamma(p+1+\lambda)} \{1 - \frac{\Gamma(p+n+1)\ \Gamma(p+1+\lambda)}{\Gamma(p+n+1+\lambda)\ \Gamma(p+1)\ B_{p+n}} z^n\}, \tag{3.11}$$

we can show that the equalities in (3.4) and (3.5) are attained for the function f(z) given by (2.2).

REMARK 7. If $B_{p+n} \geq \{\Gamma(p+n+1)\ \Gamma(p+1+\lambda)\}/\{\Gamma(p+n+1+\lambda)\ \Gamma(p+1)\}$ for $\lambda > 0$, then

$$\max \{0, \frac{\Gamma(p+1)\ |z|^{p+\lambda}}{\Gamma(p+1+\lambda)} (1 - \frac{\Gamma(p+n+1)\ \Gamma(p+1+\lambda)}{\Gamma(p+n+1+\lambda)\ \Gamma(p+1)\ B_{p+n}} |z|^n)\}$$
$$= \frac{\Gamma(p+1)|z|^{p+\lambda}}{\Gamma(p+1+\lambda)} \{1 - \frac{\Gamma(p+n+1)\ \Gamma(p+1+\lambda)}{\Gamma(p+n+1+\lambda)\ \Gamma(p+1)\ B_{p+n}} |z|^n\} .$$

Next, we derive

THEOREM 6. Let the function f(z) defined by (1.1) be in the class $A(p,n,kB_k)$ with $B_k \leq B_{k+1}$. Then we have

$$|D_z^{\lambda} f(z)| \geq \max \{0, \frac{\Gamma(p+1)|z|^{p-\lambda}}{\Gamma(p+1-\lambda)} (1 - \frac{\Gamma(p+n)\ \Gamma(p+1-\lambda)}{\Gamma(p+n+1-\lambda)\ \Gamma(p+1)\ B_{p+n}} |z|^n)\} \tag{3.12}$$

and

$$|D_z^{\lambda} f(z)| \leq \frac{\Gamma(p+1)|z|^{p-\lambda}}{\Gamma(p+1-\lambda)} \{1 + \frac{\Gamma(p+n)\ \Gamma(p+1-\lambda)}{\Gamma(p+n+1-\lambda)\ \Gamma(p+1)\ B_{p+n}} |z|^n\} \tag{3.13}$$

for $0 \leq \lambda < 1$ and $z \in$. The equalities in (3.12) and (3.13) are attained for the function f(z) given by (2.8).

PROOF. Define the function G(z) by

$$G(z) = \frac{\Gamma(p+1-\lambda)}{\Gamma(p+1)} z^{\lambda} D_z^{\lambda} f(z)$$
$$= z^p - \sum_{k=p+n}^{\infty} \frac{\Gamma(k+1)\ \Gamma(p+1-\lambda)}{\Gamma(k+1-\lambda)\ \Gamma(p+1)} a_k z^k \tag{3.14}$$

for $0 \leq \lambda < 1$. Setting

$$\psi(k) = \frac{\Gamma(k)\ \Gamma(p+1-\lambda)}{\Gamma(k+1-\lambda)\ \Gamma(p+1)} \qquad (k \geq p+n), \tag{3.15}$$

we can see that $\psi(k)$ is a decreasing function of k, that is, that

$$0 > \psi(k) \leq \psi(p+n) = \frac{\Gamma(p+n)\ \Gamma(p+1-\lambda)}{\Gamma(p+n+1-\lambda)\ \Gamma(p+1)} . \tag{3.16}$$

Consequently, it follows from (2.10) and (3.16) that

$$|G(z)| \geq \max \{0, |z|^p - \psi(p+n)|z|^{p+n} \sum_{k=p+n}^{\infty} k a_k\}$$
$$\geq \max \{0, |z|^p - \frac{\Gamma(p+n)\ \Gamma(p+1-\lambda)}{\Gamma(p+n+1-\lambda)\ \Gamma(p+1)\ B_{p+n}} |z|^{p+n}\} \tag{3.17}$$

which proves (3.12), and

$$|G(z)| \leq |z|^p + \psi(p+n)|z|^{p+n} \sum_{k=p+n}^{\infty} k a_k$$

$$\leq |z|^p + \frac{\Gamma(p+n)\ \Gamma(p+1-\lambda)}{\Gamma(p+n+1-\lambda)\ \Gamma(p+1)\ B_{p+n}} |z|^{p+n} \tag{3.18}$$

which shows (3.13).

Finally, we note that the equalities in (3.17) and (3.18) are attained for the function f(z) defined by

$$D_z^{\lambda} f(z) = \frac{\Gamma(p+1) z^{p-\lambda}}{\Gamma(p+1-\lambda)} \{1 - \frac{\Gamma(p+n)\ \Gamma(p+1-\lambda)}{\Gamma(p+n+1-\lambda)\ \Gamma(p+1)\ B_{p+n}} z^n\}. \tag{3.19}$$

This implies that the equalities in (3.12) and (3.13) are attained for the function f(z) given by (2.8).

REMARK 8. If $B_{p+n} \geq \{\Gamma(p+n)\ \Gamma(p+1-\lambda)\}/\{\Gamma(p+n+1-\lambda)\ \Gamma(p+1)\}$ for $0 \leq \lambda < 1$, then

$$\max \{0, \frac{\Gamma(p+1)|z|^{p-\lambda}}{\Gamma(p+1-\lambda)} (1 - \frac{\Gamma(p+n)\ \Gamma(p+1-\lambda)}{\Gamma(p+n+1-\lambda)\Gamma(p+1)\ B_{p+n}} |z|^n)\}$$

$$= \frac{\Gamma(p+1)|z|^{p-\lambda}}{\Gamma(p+1-\lambda)} \{1 - \frac{\Gamma(p+n)\ \Gamma(p+1-\lambda)}{\Gamma(p+n+1-\lambda)\ \Gamma(p+1)\ B_{p+n}} |z|^n\}.$$

THEOREM 7. Let the function f(z) defined by (1.1) be in the class $A(p,n,k^m B_k)$ with $B_k \leq B_{k+1}$ and $p+1 \leq m \leq p+n$. Then

$$|D_z^{\lambda} f^{(j)}(z)| \leq \frac{\left(\prod_{i=2}^{j+1} (p+n-1+i)\right) \Gamma(p+n-j)}{(p+n)^{m-1} \Gamma(p+n+1-j-\lambda)\ B_{p+n}} |z|^{p+n-j-\lambda} \tag{3.20}$$

for $0 < \lambda < 1$, $p+1 \leq j \leq m$, and $z \in U_0$, where

$$U_0 = \begin{cases} U & (p+1 \leq j \leq p+n-1) \\ U - \{0\} & (j = p+n). \end{cases}$$

PROOF. Note that

$$D_z^{\lambda} f^{(j)}(z) = - \sum_{k=p+n}^{\infty} \left(\prod_{i=1}^{j} (k+1-i) \right) \frac{\Gamma(k+1-j)}{\Gamma(k+1-j-\lambda)} a_k z^{k-j-\lambda} \tag{3.21}$$

for $p+1 \leq j \leq m$ and $0 < \lambda < 1$. Denoting

$$\theta(k) = \frac{\Gamma(k-j)}{\Gamma(k+1-j-\lambda)} \qquad (k \geq p+n),$$

we know that $\theta(k)$ is a decreasing function of k, so that

$$0 < \theta(k) \leq \theta(p+n) = \frac{\Gamma(p+n-j)}{\Gamma(p+n+1-j-\lambda)}. \tag{3.22}$$

Consequently, with the aid of Lemma 1 and (2.28), we have

$$|D_z^{\lambda} f^{(j)}(z)| \leq \frac{\Gamma(p+n-j)}{\Gamma(p+n+1-j-\lambda)} |z|^{p+n-j-\lambda} \sum_{k=p+n}^{\infty} \left(\prod_{i=1}^{j+1} (k+1-i) \right) a_k$$

$$< \frac{\Gamma(p+n-j)}{\Gamma(p+n+1-j-\lambda)} |z|^{p+n-j-\lambda} \sum_{i=1}^{j+1} \left(\frac{A_i}{(p+n)^{m-i} B_{p+n}} \right)$$

$$< \frac{\Gamma(p+n-j)}{\Gamma(p+n+1-j-\lambda)} \cdot \frac{\prod_{i=2}^{j+1} (p+n-1-i)}{(p+n)^{m-1} B_{p+n}} |z|^{p+n-j-\lambda} \tag{3.23}$$

which shows the inequality (3.20).

4. SOME PROPERTIES OF THE CLASS $A(p,n,B_k)$.

We shall give some properties of the class $A(p,n,B_k)$ consisting of functions of the form (1.1) satisfying the inequality (1.4).

THEOREM 8. $A(p,n,B_k)$ is convex set.

PROOF. We need only to prove that the function $h(z)$ defined by

$$h(z) = \delta f_1(z) + (1-\delta) f_2(z) \qquad (0 \leq \delta \leq 1) \tag{4.1}$$

is in the class $A(p,n,B_k)$ for functions $f_j(z)$ $(j=1,2)$ belonging to $A(p,n,B_k)$. Let

$$f_j(z) = z^p - \sum_{k=p+n}^{\infty} a_{k,j} z^k \qquad (a_{k,j} \geq 0; j = 1,2) \tag{4.2}$$

be in the class $A(p,n,B_k)$. Then we have

$$h(z) = z^p - \sum_{k=p+n}^{\infty} \{\delta a_{k,1} + (1-\delta) a_{k,2}\} z^k$$
$$= z^p - \sum_{k=p+n}^{\infty} c_k z^k, \tag{4.3}$$

where $c_k = \delta a_{k,1} + (1-\delta) a_{k,2}$. From this, it is easy to see that

$$\sum_{k=p+n}^{\infty} B_k c_k = \sum_{k=p+n}^{\infty} B_k \{\delta a_{k,1} + (1-\delta) a_{k,2}\}$$
$$= \delta \sum_{k=p+n}^{\infty} B_k a_{k,1} + (1-\delta) \sum_{k=p+n}^{\infty} B_k a_{k,2}$$
$$\leq \delta + (1-\delta)$$
$$= 1 \tag{4.4}$$

which implies that $h(z) \in A(p,n,B_k)$.

THEOREM 9. Let

$$f_1(z) = z^p \tag{4.5}$$

and

$$f_k(z) = z^p - \frac{1}{B_k} z^k \qquad (k \geq p+n). \tag{4.6}$$

Then $f(z)$ is in the class $A(p,n,B_k)$ if and only if it can be expressed in the form

$$f(z) = \delta_1 f_1(z) + \sum_{k=p+n}^{\infty} \delta_k f_k(z), \tag{4.7}$$

where $\delta_1 \geq 0$, $\delta_k \geq 0$ $(k \geq p+n)$, and $\sum_{k=p+n}^{\infty} \delta_k = 1 - \delta_1$.

PROOF. We assume that the function f(z) can be expressed in the form (4.7). Since

$$f(z) = (\delta_1 + \sum_{k=p+n}^{\infty} \delta_k) z^p - \sum_{k=p+n}^{\infty} \frac{\delta_k}{B_k} z^k$$

$$= z^p - \sum_{k=p+n}^{\infty} \frac{\delta_k}{B_k} z^k$$

$$= z^p - \sum_{k=p+n}^{\infty} d_k z^k , \tag{4.8}$$

we observe that

$$\sum_{k=p+n}^{\infty} B_k d_k = \sum_{k=p+n}^{\infty} \delta_k = 1 - \delta_1 \leq 1, \tag{4.9}$$

that is, that $f(z) \in A(p,n,B_k)$.

Conversely, assume that the function f(z) defined by (1.1) is in the class $A(p,n,B_k)$. Then, it follows that

$$a_k \leq \frac{1}{B_k} \qquad (k \geq p+n). \tag{4.10}$$

Therefore, we may put

$$\delta_k = B_k a_k \qquad (k \geq p+n)$$

and

$$\delta_1 = 1 - \sum_{k=p+n}^{\infty} \delta_k .$$

Thus we prove that

$$f(z) = z^p - \sum_{k=p+n}^{\infty} a_k z^k$$

$$= \delta_1 f_1(z) + \sum_{k=p+n}^{\infty} \frac{\delta_k}{B_k} z^k$$

$$= \delta_1 f_1(z) + \sum_{k=p+n}^{\infty} \delta_k f_k(z) . \tag{4.11}$$

This completes the assertion of Theorem 9.

By virtue of Theorem 8 and Theorem 9, we have

COROLLARY 1. The extreme points of $A(p,n,B_k)$ are $f_1(z)$ and $f_k(z)$ $(k \geq p+n)$ defined in Theorem 9.

Next, we prove

THEOREM 10. Let $f_j(z)$ (j=1,2) defined by (4.2) be in the class $A(p,n,B_{k,j})$.

Then the function h(z) defined by

$$h(z) = z^p - \sum_{k=p+n}^{\infty} a_{k,1}a_{k,2}z^k \tag{4.12}$$

is in the class $A(p,n,B_{k,3})$, where $B_{k,3} \leq B_{k,1}B_{k,2}$.

PROOF. We need to prove that

$$\sum_{k=p+n}^{\infty} B_{k,3}a_{k,1}a_{k,2} \leq 1 \tag{4.13}$$

for $B_{k,3} \leq B_{k,1}B_{k,2}$. Since

$$\sum_{k=p+n}^{\infty} B_{k,j}a_{k,j} \leq 1 \qquad (j = 1,2), \tag{4.14}$$

by using the Cauchy-Schwarz inequality, we have

$$\sum_{k=p+n}^{\infty} \sqrt{B_{k,1}B_{k,2}} \quad \sqrt{a_{k,1}a_{k,2}} \leq 1. \tag{4.15}$$

Hence, if

$$B_{k,3}a_{k,1}a_{k,2} \leq \sqrt{B_{k,1}B_{k,2}}\sqrt{a_{k,1}a_{k,2}} \qquad (k \geq p+n), \tag{4.16}$$

or

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{\sqrt{B_{k,1}B_{k,2}}}{B_{k,3}} \qquad (k \geq p+n), \tag{4.17}$$

then the inequality (4.13) is satisfied. Since

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{1}{\sqrt{B_{k,1}B_{k,2}}} \qquad (k \geq p+n) \tag{4.18}$$

by means of (4.15), we can show that if

$$\frac{1}{\sqrt{B_{k,1}B_{k,2}}} \leq \frac{\sqrt{B_{k,1}B_{k,2}}}{B_{k,3}} \qquad (k \geq p+n), \tag{4.19}$$

that is, if $B_{k,3} \leq B_{k,1}B_{k,2} (k \geq p+n)$, then (4.13) is satisfied. Thus we have Theorem 10.

Finally, we derive

THEOREM 11. Let $f_j(z)$ $(j=1,2,\ldots,m)$ defined by (4.2) be in the class $A(p,n,B_k)$. Then the function

$$h(z) = z^p - \sum_{k=p+n}^{\infty} \left(\sum_{j=1}^{m} a_{k,j}^2 \right) z^k \tag{4.20}$$

is in the class $A(p,n,C_k)$, where $C_k \leq B_k^2/m$.

PROOF. It is sufficient to show that

$$\sum_{k=p+n}^{\infty} C_k \left(\sum_{j=1}^{m} a_{k,j}^2 \right) \leq 1 \tag{4.21}$$

for $C_k \leq B_k^2/m$. Note that, for $f_j(z) \in A(p,n,B_k)$ $(j=1,2,\dots,m)$,

$$\sum_{k=p+n}^{\infty} B_k^2 a_{k,j}^2 \leq \left(\sum_{k=p+n}^{\infty} B_k a_{k,j} \right)^2 \leq 1 \quad (j = 1,2\dots,m). \tag{4.22}$$

It follows from (4.22) that

$$\frac{1}{m} \sum_{k=p+n}^{\infty} B_k^2 \left(\sum_{j=1}^{m} a_{k,j}^2 \right) \leq 1. \tag{4.23}$$

Consequently, we have

$$\sum_{k=p+n}^{\infty} C_k \left(\sum_{j=1}^{m} a_{k,j}^2 \right) \leq \frac{1}{m} \sum_{k=p+n}^{\infty} B_k^2 \left(\sum_{j=1}^{m} a_{k,j}^2 \right) \leq 1 \tag{4.24}$$

for $C_k \leq B_k^2/m$ which completes the proof of Theorem 11.

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