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RESEARCH NOTES

A CONVEX OPERATOR FUNCTION

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ABSTRACT. It is shown that inversion is a convex function on the set of strictly positive elements of a C^*-algebra.

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1. INTRODUCTION.

A real-valued function f defined on a real interval I is said to be convex if

$$f(\lambda s + (1-\lambda)t) \leq \lambda f(s) + (1-\lambda) f(t)$$

for $s,t \in I$ and $0 \leq \lambda \leq 1$. Convex functions play a fundamental role in the study of the Lebesgue L^p spaces [1], [2]. Geometrically, a function f is convex if the chord joining the points $(s,f(s))$ and $(t,f(t))$ lies above the graph of f. An interesting example of a convex function is the function $f(t) = t^{-1}$, $t \in I = (0,\infty)$. Thus inversion is a convex function on the set of positive reals. The notion of convexity has been generalized to functions with domain and range more general than reals. For instance, through a diagonalization process it is shown in [3] that inversion is a convex function on the set of positive-definite real symmetric matrices. In this note we will show that this result holds in a C^*-algebra. More precisely, we use Banach algebra techniques to show that inversion is a convex function on the set of strictly positive elements of a C^*-algebra.

2. PRELIMINARIES.

Throughout this article $\mathcal{A}$ will denote a complex C^*-algebra with identity e. An element $x \in \mathcal{A}$ is said to be self-adjoint if $x^* = x$, where x^* is the adjoint of x. A self-adjoint element x is said to be non-negative, in notation $x \geq 0$, if its spectrum $\sigma(x)$ lies in the interval $[0,\infty)$. For self-adjoint elements x and y, we write $x \leq y$ if $y - x \geq 0$. An element x will be termed strictly positive if it is non-negative and invertible. Thus x is strictly positive if x is self-adjoint and $\sigma(x)$ lies in the interval $(0,\infty)$. If x is an invertible element then we use x^{-1} to denote its inverse.

A subalgebra $\mathcal{B}$ of $\mathcal{A}$ is said to be self-adjoint if $x \in \mathcal{B}$ implies $x^* \in \mathcal{B}$. The main tools we need to establish our result are:

(A) If $\mathcal{B}$ is a closed self-adjoint subalgebra of $\mathcal{A}$ and $x \in \mathcal{B}$, then $\sigma_{\mathcal{B}}(x) = \sigma_{\mathcal{A}}(x)$. Here $\sigma_{\mathcal{B}}(x)$ and $\sigma_{\mathcal{A}}(x)$ denote the spectra of x relative to $\mathcal{B}$ and $\mathcal{A}$, respectively.

(B) If $\mathfrak{B}$ is a commutative Banach algebra and $x \in \mathfrak{B}$ then $\sigma_{\mathfrak{B}}(x) = \{\varphi(x) \mid \varphi$ a complex homomorphism on $\mathfrak{B}\}$.

Proofs of (A) and (B) may be found, for example, in [4].

3. MAIN RESULT.

LEMMA: If w is a strictly positive element of $\mathcal{A}$, then

$$[\lambda e + (1-\lambda)w]^{-1} \leq \lambda e + (1-\lambda)w^{-1} \text{ for } 0 \leq \lambda \leq 1.$$

PROOF: Let $\mathfrak{B}$ be the closed subalgebra generated by w and e. Since w is self-adjoint, $\mathfrak{B}$ is self-adjoint and commutative. Clearly w and $\lambda e + (1-\lambda)w$ are elements of $\mathfrak{B}$. Since these elements are invertible in $\mathcal{A}$, $u = [\lambda e + (1-\lambda)w]^{-1}$ and $v = \lambda e + (1-\lambda)w^{-1}$ are elements of $\mathfrak{B}$ by (A). Our goal is to show that $\sigma_{\mathfrak{B}}(v-u)$ lies in $[0,\infty)$. In view of (B) it suffices to show that $\varphi(u) \leq \varphi(v)$ for complex homomorphisms φ on $\mathfrak{B}$. Since $\varphi(u) = [\lambda + (1-\lambda)\varphi(w)]^{-1}$ and $\varphi(v) = \lambda + (1-\lambda)(\varphi(w))^{-1}$, the result follows from the fact that $f(t) = t^{-1}$ is a convex function on $(0,\infty)$.

THEOREM: If x and y are strictly positive elements of $\mathcal{A}$, then

$$[\lambda x + (1-\lambda)y]^{-1} \leq \lambda x^{-1} + (1-\lambda)y^{-1} \text{ for } 0 \leq \lambda \leq 1.$$

PROOF: First we recall that if p and q are self-adjoint elements of $\mathcal{A}$ with $p \leq q$, then $r^*pr \leq r^*qr$ for any $r \in \mathcal{A}$. This fact from C^*-algebra theory will be used twice in the proof.

Now, since x is strictly positive, it possesses a unique strictly positive square root, say z, in $\mathcal{A}$. Then $w = z^{-1}yz^{-1}$ is strictly positive. By the lemma, we have

$$[\lambda e + (1-\lambda)w]^{-1} \leq \lambda e + (1-\lambda)w^{-1}$$

Thus

$$z^{-1}[\lambda e + (1-\lambda)w]^{-1}z^{-1} \leq z^{-1}[\lambda e + (1-\lambda)w^{-1}]z^{-1}$$

This in turn gives

$$[\lambda x + (1-\lambda)y]^{-1} \leq \lambda x^{-1} + (1-\lambda)y^{-1}$$

The proof is thus complete.

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