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INTEGRALS OF OPERATOR-VALUED FUNCTIONS

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ABSTRACT. Mikusinski-type expansions of operator-valued functions are discussed in some detail. As a natural part of the development, a "kernel" concept for operators is proposed and an elaborate system of convolution quotients in one and two variables is obtained.

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1. INTRODUCTION.

Jan Mikusinski has presented [1] a very simple scheme for the development of general integrals. In the case of the Lebesgue integral on the real line R, for example, it consists of selecting real numbers λ_n and brick functions f_n (characteristic functions of finite intervals) satisfying

$$\sum |\lambda_n| \int f_n < \infty \qquad \left(\int f_n = \text{length of the carrier of } f_n \right), \tag{1.1}$$

and then summing the series

$$f(x) = \sum \lambda_n f_n(x) \tag{1.2}$$

at those points x for which the series converges absolutely. Any real-valued function f satisfying (1.2) is Lebesgue integrable over R and its Lebesgue integral is given simply by the sum of the integrals,

$$\int f = \sum \lambda_n \int f_n. \tag{1.3}$$

Perhaps surprisingly, it turns out that every Lebesgue class has such an expansion. The entire Lebesgue theory can be based simply upon the concept of absolutely convergent series of numbers! Mikusinski has introduced the notation

$$f \cong \sum \lambda_n f_n$$

to indicate the validity of such an expansion.

The extension of this simple scheme to the Lebesgue integrals for real-valued functions on higher dimensional real spaces and to the Bochner integrals for vector-valued functions (where the λ_n are in a Banach space and $|\lambda_n|$ denotes the norms of the λ_n) is straightforward and entails no serious additional complications [1].

In this paper we examine the situation where the function values lie in certain generalized functions spaces. Specifically, we discuss distribution-valued functions and Mikusinski operator-valued functions, with the major emphasis being placed upon the latter. We are led to the kernel theorem for distributions and, consequently, propose a "kernel" concept for operators. A general arithmetical system of convolution quotients and operations evolves naturally from this rather formal program; however, there are no serious theorems to be found here - only definitions, explanations and examples. (For serious theorems, see Mikusinski's book [1].)

2. DISTRIBUTION-VALUED FUNCTIONS.

Let $\mathcal{D}'$ denote the space of distributions on R and let $\mathcal{D}$ denote the space of infinitely differentiable test functions of compact support.

If $\lambda_n \in \mathcal{D}'$ and if f_n are brick functions, then (in analogy with the Lebesgue case) we write

$$f \cong \sum \lambda_n f_n, \quad \text{if for each } \phi \in \mathcal{D},$$

$$\sum |\langle\lambda_n,\phi\rangle| \int f_n < \infty \tag{2.1}$$

and

$$\langle f,\phi\rangle\ (x) = \sum \langle\lambda_n,\phi\rangle\ f_n(x) \tag{2.2}$$

at those points x for which the series converges absolutely. In such a case, it is clear that

$$F = \sum \lambda_n \int f_n \quad \text{converges in } \mathcal{D}', \tag{2.3}$$

(i.e. $\langle F,\phi\rangle = \sum \langle\lambda_n,\phi\rangle \int f_n$ for each $\phi \in \mathcal{D}$.)

So $F \in \mathcal{D}'$. But what is f itself? By (2.1), (2.2) here, one has for each $\phi \in \mathcal{D}$,

$$\langle f,\phi\rangle \cong \sum \langle\lambda_n,\phi\rangle f_n$$

in the original Lebesgue sense; so f maps $\mathcal{D}$ into the Lebesgue space $\mathcal{L}$. It is not difficult to see that f is linear and continuous (because of (2.1)). Hence it extends, by the kernel theorem, to a distribution of two variables (which we denote by t and x) on the test function space $\mathcal{D}_{t,x} = \mathcal{D}_t \otimes \mathcal{D}_x$ of two variables. If we retain the notation f for the extended (kernel) distribution, then f applied to the special product $\phi(t)\psi(x)$, with $\phi(t) \in \mathcal{D}_t$ and $\psi(x) \in \mathcal{D}_x$, is simply

$$\langle f\ (t,x),\phi(t)\psi(x)\rangle = \sum \langle\lambda_n,\phi\rangle\langle f_n,\psi\rangle = \sum \langle\lambda_n,\phi\rangle \int f_n\psi.$$

(There are, of course, other test functions in $\mathcal{D}_{t,x}$.) This extended distribution is semi-regular in t, since (as in (2.2))

$$<f(t,x),\phi(t)> = \sum <\lambda_n,\phi> f_n(x)$$

is an ordinary (Lebesgue integrable) function, but in general is not semi-regular in x, since

$$<f(t,x),\psi(x)> = \sum \lambda_n \int f_n\psi$$

is only a distribution in $\mathcal{D}_t'$ (as in (2.3)). In this context (2.3) becomes

$$F = \sum \lambda_n \int f_n = \int f(t,x)dx.$$

We shall encounter an analogus situation in the following section. However, before taking up the case of operator-valued functions let us make a couple of observations concerning the brick functions f_n. Condition (2.1) is a requirement only on the lengths of the carriers, and so in (2.2) the carriers themselves can be distributed about the real line R arbitrarily, always producing an integrable function. However, once the locations of the carriers are selected, then they remain the same collection of brick functions in (2.2) is ϕ varies in $\mathcal{L}$. So the kernel f(t,x) which is integrable with respect to x appears to be rather special in this category. (See section 4, where the analogus situation is seen not to be the case for operators.)

3. OPERATOR-VALUED FUNCTIONS.

Let $\mathcal{M}_t$ denote the field of Mikusinski operators on the half line $t \geq 0$, and let $\mathcal{C}_t$ denote the convolution ring of continuous functions on this half line. ($\mathcal{M}_t$ is, of course, the algebraic field of equivalence classes of convolution quotients of $\mathcal{C}_t$.)

If $\lambda_n \in \mathcal{M}_t$ and if f_n are brick functions, then we write

$$f \cong \sum \lambda_n f_n,$$

if for some nonzero $\sigma(t) \in \mathcal{C}_t$ we have $\sigma\lambda_n(t) \in \mathcal{C}_t$ for all n,

$$\sum |\sigma\lambda_n(t)| \int f_n \text{ converges in } \mathcal{C}_t \tag{3.1}$$

(i.e. uniformly on compact sets) and for all $t \geq 0$,

$$\sigma f(x) = \sum \sigma\lambda_n(t) f_n(x) \tag{3.2}$$

at those points x for which the series converges absolutely. In such a case, it follows that

$$F = \sum \lambda_n \int f_n \text{ converges in } \mathcal{M}_t \text{ (type I convergence)} \tag{3.3}$$

(i.e. for some nonzero $\sigma(t) \in \mathcal{C}_t$, $\sigma F(t) = \sum \sigma\lambda_n(t) \int f_n$ converges in $\mathcal{C}_t$.)

In this situation f itself can be interpreted as a mapping similar to what occurs in the distributional case. Indeed, the collection $I_f = \{\sigma \in \mathcal{C}_t \mid (3.1) \text{ holds}\}$ forms a $\mathcal{C}_t$-ideal and for σ in this ideal, (3.1) and (3.2) imply that for each $t \geq 0$,

$$\sigma f \cong \sum \sigma\lambda_n(t) f_n$$

in the Lebesgue sense (and that the sum of the series is a continuous function of t). Thus f maps the ideal I_f into $\mathcal{C}_t \times \mathcal{C}_x$ and is linear, multiplicative in t (commutes with convolution involving elements of $\mathcal{C}_t$) and is continuous. We can interpret f as an operator-valued function of $x \in R$, which is integrable with respect to x and whose integral over R is given by (3.3). In the following section we shall give a reinterpretation of this situation using the more traditional setting of convolution fractions.

4. SEMI-OPERATORS.

Let $\mathcal{H} = \{h(t,x) \mid h$ is locally integrable in the two variables, supported on the half space $t \geq 0$ and $\int h(t,x)dx$ exists for almost every $t \geq 0\}$. Actually, we need to work here and elsewhere with Lebesgue equivalence classes of such functions but will not introduce additional notation for such purposes. Note that $\mathcal{C}_t \times \mathcal{C}_x \subset \mathcal{H}$.

In $\mathcal{H}$ we introduce the natural convolution (in two variables)

$$h*k(t,x) = \int_0^t \int_{-\infty}^{\infty} h(\tau,y)k(t-\tau,x-y)dyd\tau$$

and addition + (pointwise) to form a ring with many divisors of zero. However, it is readily seen that a mixed convolution $\sigma(t)*h(t,x)$ in the t variable only, with $\sigma \in \mathcal{C}_t$ and $h \in \mathcal{H}$, can result in the zero of $\mathcal{H}$ only if at least one of the factors is the zero of its respective ring. Hence we can form meaningful convolution fractions of the form h/σ ($h \in \mathcal{H}$ and nonzero $\sigma \in \mathcal{C}_t$) with equivalency, convolution product and addition defined in the expected way. The equivalence classes we will call <u>semi-operators</u> and we note that they form a ring.

If $f \cong \sum \lambda_n f_n$ is an expansion, as in the previous section, then (3.1), (3.2) and (3.3) mean that f is the semi-operator h/σ, where $h = \sigma f = \sum \sigma\lambda_n f_n$ (i.e. $h(t,x) = \sum \sigma\lambda_n(t)f_n(x)$ in $\mathcal{H}$), while $F = \sum \lambda_n \int f_n = \int f\,dx = \int h(t,x)dx/\sigma(t) \in \mathcal{M}_t$, the field of Mikusinski operators in the t variable.

An interesting example of a semi-operator can be constructed using an infinite series $\sum a_n S^n$ in the differentiation operator S. Boehme [2] has shown that such series converges in $\mathcal{M}_t$ (type I) if the sequence of numbers a_n is appropriate. (He gave necessary and sufficient conditions for this to happen.) His proof showed absolute convergence, as in (3.1), and hence if we select brick functions f_n so that their carrier lengths satisfy $\int f_n = |a_n|$ for such an appropriate sequence, then $f \cong \sum S^n f_n$ becomes a Mikusinski-type expansion of a semi-operator, where $\int f = \sum |a_n| S^n$.

In order to pursue the analogy with distribution theory and obtain some sort of kernel operator associated with a semi-operator it becomes necessary to treat the two variables symmetrically. We do this in the next section and obtain not only kernels but a general arithmetical system of fractions which may be of some independent interest.

5. AN ARITHMETIC OF FRACTIONS, KERNEL OPERATORS, SELF CONVOLUTION.

For our purposes in this section we introduce still another space of functions, namely,

$\mathcal{H}^+ = \{h(t,x) \mid$ h is locally integrable and supported in the quarter space $t \geq 0,\ x \geq 0\}$.

This is a substitute for the space $\mathcal{F}$ of section 4 where we have further limited the supports of our functions but we do not impose any global integrability conditions. The latter is unnecessary since convolution in two variables is guaranteed by the support restrictions. This space (of Lebesgue equivalence classes) becomes a ring without divisors of zero under convolution and addition, and its quotient space is isomorphic with (and shall be identified with) the Mikusinski operator field $\mathcal{M}_{t,x}$ in two variables.

Now one can form two rings, say $\mathcal{F}_t$ and $\mathcal{F}_x$, of semi-operators of the two forms $\frac{h(t,x)}{\sigma(t)}$ and $\frac{k(t,x)}{\psi(x)}$, with h,k $\mathcal{H}^+$ and nonzero $\sigma(t) \in \mathcal{C}_t$, nonzero $\psi(x) \in \mathcal{C}_x$. Of course these rings are isomorphic but we shall keep them distinguished here by indicating the single variables in the denominators. It is quite natural (and pertinent) to define another ring $\mathcal{K}$ of fractions of the special form $\frac{h(t,x)}{\sigma(t)\psi(x)}$, which we will call <u>kernel</u> operators. They are merely Mikusinski operators in $\mathcal{M}_{t,x}$ with special denominators (where the variables are separated).

In addition to these three rings of fractions we wish to consider all fractions constructed without zero denominators using any combination of two functions from the three rings $\mathcal{H}^+, \mathcal{C}_t, \mathcal{C}_x$. Examples are the ten forms

$$\mathcal{F}: \frac{h(t,x)}{k(t,x)}, \frac{h(t,x)}{\sigma(t)\psi(x)}, \frac{h(t,x)}{\sigma(t)}, \frac{k(t,x)}{\psi(x)}, \frac{\phi(t)}{\sigma(t)}, \frac{\omega(x)}{\psi(x)}, \frac{\phi(t)}{h(t,x)}, \frac{\psi(x)}{k(t,x)}, \frac{\omega(x)}{\sigma(t)}, \frac{\phi(t)}{\psi(x)}$$

with h,k $\in \mathcal{H}^+$ and $\sigma,\phi \in \mathcal{C}_t$, $\psi,\omega \in \mathcal{C}_x$.

The first six of these belong, respectively, to the convolution rings $\mathcal{M}_{t,x}, \mathcal{K}, \mathcal{F}_t, \mathcal{F}_x, \mathcal{M}_t, \mathcal{M}_x$ identified earlier while the last two are formal numerical fractions. These and the other two forms can be thought of as belonging to four other multiplicative semigroups, which we might label $\mathcal{F}^t, \mathcal{F}^x, \mathcal{N}_t^x, \mathcal{N}_x^t$, for the sake of completeness.

Any two of these ten types of fractions can be multiplied simply by multiplying the numerator and denominator functions separately to form the product fraction, which will again be one of these ten types. Multiplication of two functions <u>means</u> convolution whenever variables are repeated in t, x or both variables. When no variables are repeated then multiplication means ordinary numerical pointwise multiplication. Equivalency of fractions is then the expected one (cross multiplication) based upon this general rule for multiplication of functions, and addition of fractions is also defined in the expected way based upon this rule for multiplication of functions. Because of the latter the expected distributivity property of multiplication with respect to addition holds. Moreover, the expected associativity (and commutativity) properties hold, as may be verified directly. But we note also that each of the nonzero fractions in this collection has an inverse which is again in the collection! In short, the collection $\mathcal{F}$ of (equivalence classes of) the ten forms of fractions becomes a field, provided multiplication is appropriately interpreted. (The three rings of functions $\mathcal{H}^+$, $\mathcal{C}_t$, $\mathcal{C}_x$ can also be joined to the field.) Actually this field $\mathcal{F}$ is isometric to the subfield $\mathcal{M}_{t,x}$ of Mikusinski operators itself. An isomorphism is simply the mapping $f \to f\,\frac{\ell}{\ell}$, where ℓ is any fixed nonzero function in $\mathcal{H}^+$. However, the structural differences among these various fractional forms are masked by such an isomorphism (i.e. the fractional forms reveal some of the interesting sub structures of the Mikusinski field in two variables). There is no difficulty in generalizing this formal construction of fields of fractions which involve more than two variables. The essential step is to just interpret multiplication of functions properly.

If $f = \frac{h(t,x)}{\sigma(t)}$ is a semi-operator in $\mathcal{F}_t$, then the fraction $\overline{f} = \frac{h(t,x)*\psi(x)}{\sigma(t)\psi(x)} = \frac{\ell(t,x)}{\sigma(t)\psi(x)}$ (for any nonzero $\psi \in \mathcal{C}_x$) will be called the <u>kernel operator associated</u> with f. In this case, where $\sigma\,\overline{f} = \sigma f = \frac{\ell}{\psi} = h$ is an ordinary function (in $\mathcal{C}^+$), the kernel is said to be <u>semi-regular</u> in t. In general $\psi\overline{f} = \psi f = \frac{\ell}{\sigma}$ is not an ordinary function for any nonzero $\psi \in \mathcal{C}_x$, so that $\overline{f}$ is not semi-regular in x.

In the present setting a Mikusinski-type expansion (in x, say) requires that the brick functions used have carriers restricted to the half line $x \geq 0$. Then the semi-operator $f \in \mathcal{F}_t$ has an expansion $f \cong \sum \lambda_n f_n$, if $\lambda_n \in \mathcal{M}_t$ and f_n are brick functions on the half line $x \geq 0$ such that for some nonzero $\sigma(t) \in \mathcal{C}_t$ we have $\sigma\lambda_n(t) \in \mathcal{C}_t$ for all n,

$$\sum |\sigma\lambda_n(t)| \int f_n \text{ converges in } \mathcal{C}_t, \tag{5.1}$$

and for all $t \geq 0$

$$\sigma f(x) = \sum \sigma\lambda_n(t) f_n(x) \tag{5.2}$$

at those points x for which the series converges absolutely. In such a case, $f = \frac{h(t,x)}{\sigma(t)}$, where h(t,x) is given on the right in (5.2), and we have

$$F = \sum \lambda_n \int f_n \text{ converges in } \mathcal{M}_t \tag{5.3}$$

with $F = \frac{\int h(t,x)dx}{\sigma(t)}$. Here, of course, the kernel operator $\overline{f} = \frac{h(t,x)*\psi(x)}{\sigma(t)\psi(x)} = \frac{\ell(t,x)}{\sigma(t)\psi(x)}$ is semi-regular in t (i.e. $\sigma\overline{f} = \sigma f = h = \sum \sigma\lambda_n f_n$ is a function). However, the product $\psi\overline{f} = \psi f = \sum \lambda_n \int f_n \psi$ is in general only an operator in $\mathcal{M}_t$, so $\overline{f}$ is not semi-regular in x.

Similar observations concerning an expansion of a semi-operator from $\mathcal{F}_x$ could be made simply upon interchanging the roles of the two variables. Perhaps it is unnecessary to do so explicitly.

Finally, because we deal with two variables it is of interest here to introduce a third natural operation, called self convolution which can be applied to all fractions in the field $\mathcal{F}$. First for functions - the self convolution of a function of one variable k(x) will simply be the function k(x) itself (or, if desired, a shift to the other variable k(t)), while for a function of two variables the self convolution of h(t,x) will be the function of one variable given by

$$h_*(x) = \int_0^x h(t,x-t)dt = \int_0^x h(x-t,t)dt$$

(or, if desired, the same function with x replaced by the t variable, $h_*(t)$). When h has separated variables, say $h(t,x) = h_1(t)h_2(x)$, self convolution h_* becomes just ordinary convolution $h_1 * h_2$ of the two factors, hence the name. Moreover, self convolution in this circumstance is distributive with respect to ordinary convolution, since if $g(t,x) = g_1(t)g_2(x)$, then $h_* * g_* = h_1 * h_2 * g_1 * g_2 = (h*g)_*$. In fact, this distributivity property holds quite generally for all locally integrable functions (as will be shown in section 6) regardless of the number of variables of the factors or which of the variables appear in the factors.

The self convolution f_* of a fraction in $\mathcal{F}$ is then defined as that fraction obtained from the self convolution of the numerator and denominator functions individually, provided the denominator does not vanish. This last can occur only for a function of two variables. Self convolution of fractions is distributive with respect to multiplication as well as addition, that is, the equations

$$f_* \cdot g_* = (f \cdot g)_* \quad , \quad f_* + g_* = (f + g)_*$$

hold in the field $\mathcal{F}$ (when denominators are nonzero). Actually, we have <u>two</u> kinds of self convolution - one where the resulting variable is x and one where the resulting variable is t. Both exhibit the above distributivity properties.

For a Mikusinski-type expansion $f \cong \sum \lambda_n f_n = \frac{h(t,x)}{\sigma(t)}$ we might write (rather naturally)

$$f_* \cong \sum \lambda_n * f_n, \text{ where } f_*(t) = \frac{h_*(t)}{\sigma(t)}, \; \lambda_n * f_n = \frac{\sigma\lambda_n * f_n}{\sigma}, \text{ and where this}$$

series converges (uniformly on compact sets) because of (5.1).

We shall conclude this section with an example using convergent infinite series in differentiation operators of the type considered in section 4. Let $f \cong \sum S_t^n f_n(x)$ and $g \cong \sum S_x^m g_m(t)$ be two convergent Mikusinski-type expansions of semi-operators in $\mathcal{F}_t$ and $\mathcal{F}_x$, respectively. Here f_n and g_m are brick functions on $x \geq 0$ and $t \geq 0$, and S_x and S_t denote the derivative operators in the indicated variables. Then

$$f = \frac{\sum \sigma^{(n)}(t) f_n(x)}{\sigma(t)} \quad \text{and} \quad g = \sum \frac{\psi^{(m)}(x) g_m(t)}{\psi(x)}$$

for certain infinitely differentiable σ and ψ, where the exponents denote ordinary differentiation. Thus,

$$f \cdot g = \frac{\sum\sum [\sigma^{(n)} * g_m(t)] \cdot [\psi^{(m)} * f_n(x)]}{\sigma(t)\psi(x)}$$

is the product of these two fractions in the field $\mathcal{F}$. However, we can also consider another, even more interesting combination of these two semi-operators in the form of a convergent Mikusinski-type expansion in two sets of two variables, namely the tensor product operator

$$p(\tau,y;t,x) \cong \sum\sum S_\tau^n S_y^m g_m(t) f_n(x) \; .$$

The products $g_m(t)f_n(x)$ form brick functions in two variables (t,x) and the products $S_\tau^n S_y^m$ are Mikusinski operators in two variables (τ,y). Then

$$\sigma\psi p(\tau,y;t,x) = \sum\sum \sigma^{(n)}(\tau)\psi^{(m)}(y)g_m(t)f_n(x),$$

for suitable infinitely differentiable σ and ψ. Hence (by self convoluting in two sets of two variable each) we obtain

$$\int_0^x \int_0^t \sigma\psi p(\tau,y;\ t-\tau,x-y)d\tau dy = \sum\sum [\sigma^{(n)}*g_m(t)]\cdot[\psi^{(m)}*f_n(x)]$$

which, as can be seen from the above, is also equal to the function $\sigma\psi f\cdot g(t,x)$. This means that the field product $f\cdot g(t,x)$ in two variables (t,x) is the self convolution $p_*(t,x)$ of the multidimensional semi-operator $p(\tau,y;t,x)$ in two sets, (τ,y) and (t,x) of two variables each.

6. EXPANSIONS FOR LOCALLY INTEGRABLE FUNCTIONS.

If $h \in \mathcal{H}^+$, then $h = h(t,x)$ is integrable over the square $0 \leq t \leq N$, $0 \leq x \leq N$ for each natural number N. Because we deal with absolutely convergent series and because these squares cover the quarter space $t \geq 0$, $x \geq 0$, there exists a Mikusinski-type (two-dimensional) expansion for h of the form

$$h(t,x) \cong \sum\sum \lambda_{mn}g_m(t)f_n(x),$$

where the λ_{mn} are real numbers and g_m, f_n are brick functions supported on $t \geq 0$, $x \geq 0$, respectively. In this situation we have

$$\sum\sum |\lambda_{mn}| \int_0^N g_m \int_0^N f_n < \infty, \text{ for each N} \tag{6.1}$$

and

$$h(t,x) = \sum\sum \lambda_{mn}g_m(t)f_n(x) \tag{6.2}$$

at those points (t,x) for which the series converges absolutely. This results in the integral

$$\int_0^N \int_0^N h\,dt\,dx = \sum\sum \lambda_{mn} \int_0^N g_m \int_0^N f_n, \quad \text{for each N.} \tag{6.3}$$

Note that the series in (6.2) converges to h almost everywhere in the <u>entire</u> quarter space $t \geq 0$, $x \geq 0$, so we can identify this one series with h throughout.

One application of this expansion result is the proof that self convolution is distributive with respect to ordinary convolution. For if also,

$$k(t,x) \cong \sum\sum \mu_{ij}k_i(t)h_j(x),$$

then

$$h(t,x)*k(t,x) = \sum\sum\sum\sum \lambda_{mn}\mu_{ij}[g_m*k_i(t)]\cdot[f_n*h_j(x)],$$

so that

$$(h * k)_*(x) = \sum\sum\sum\sum \lambda_{mn}\mu_{ij}g_m*k_i*f_n*h_j(x).$$

While on the other hand,

$$h_*(x) = \sum\sum \lambda_{mn}g_m*f_n(x), \qquad k_*(x) = \sum\sum \mu_{ij}k_i*h_j(x),$$

and so $h_* * k_*(x)$ is the same fourth order sum with four convolutions in each term. A similar argument can be given for all the other cases considered in section 5.

Other interesting applications are to Mikusinski-type expansions of arbitrary kernels and semi-operators. Indeed we have immediately for kernels

$$\frac{h(t,x)}{\sigma(t)\psi(x)} \cong \sum\sum \lambda_{mn} \frac{g_m(t)}{\sigma(t)} \frac{f_n(x)}{\psi(x)} ,$$

where

$$\frac{g_m(t)}{\sigma(t)} \in \mathcal{M}_t \text{ and } \frac{f_n(x)}{\psi(x)} \in \mathcal{M}'_x ,$$

and for semi-operators in $\mathcal{F}_t$,

$$\frac{h(t,x)}{\sigma(t)} \cong \sum_n \left(\sum_m \frac{\lambda_{mn} g_m(t)}{\sigma(t)} \right) f_n(x) ,$$

where $\sum_m \frac{\lambda_{mn} g_m(t)}{\sigma(t)} \in \mathcal{M}_t$, and for semi-operators in $\mathcal{F}_x$,

$$\frac{h(t,x)}{\psi(x)} \cong \sum_m \left(\sum_n \frac{\lambda_{mn} f_n(x)}{\psi(x)} \right) g_m(t),$$

where

$$\sum_n \frac{\lambda_{mn} f_n(x)}{\psi(x)} \in \mathcal{M}_x .$$

These expansions are of the local type (on squares, as in (6.1), though valid throughout, as in (6.2)) and not necessarily the same as those considered in section 5. The later two do become the former versions, however, when h(t,x) is integrable over the half line $x \geq 0$ or $t \geq 0$, respectively.

The above Mikusinski-type expansion result for locally integrable functions supported on the quarter space $t \geq 0$, $x \geq 0$, is easily extended to apply to arbitrary locally integrable functions without suppost restrictions. In particular, the semi-operator expansions of sections 3 and 4 can be shown to encompass all the cases where the operator-valued functions of $x \in R$ are integrable over R.

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