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ON ASSOCIATIVE COPULAS UNIFORMLY CLOSE

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ABSTRACT. Associative copulas uniformly close are studied with detail. Some classical results of Ulam and Hyers as well as the representation theorem for associative functions play a fundamental role.

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1. INTRODUCTION.

Associative operations on the positive real half-line which are uniformly close have recently been characterized in [1]. In the present paper we turn to the study of associative copulas which are uniformly close. Copulas play a fundamental role in the theory of probabilistic metric spaces (see Menger [2] and Schweizer-Sklar [3]), in probability theory (cf. Fréchet [4], Sklar [5] and [6]) as well as in the study of nonparametric measures of dependence for random variables (see Schweizer-Wolff [7]). We begin with some preliminary notions. From now on I will denote the closed unit interval [0,1].

DEFINITION 1.1. A binary operation T on I is called an Archimedean t-norm if T is continuous, associative, commutative, nondecreasing in each place, 1 is a unit and $T(x,x) < x$ whenever x is in (0,1).

The following result due to J. Aczél ([8]) and C.H. Ling ([9]) gives the general representation for Archimedean t-norms.

THEOREM 1.1. A binary operation T on I is an Archimedean t-norm if and only if there exists a continuous and strictly decreasing function $f:I \to [0,\infty]$ such that $f(1) = 0$ and $T(x,y) = f^{(-1)}(f(x)+f(y))$, where $f^{(-1)}$ stands for the function

$$f^{(-1)}(x) := f^{-1}(\min(x,f(0))), \quad x \in [0,\infty] . \tag{1.1}$$

In the case where $f(0) = \infty$, we have simply $f^{(-1)} = f^{-1}$ and the corresponding operation T is called strict; otherwise, T is said to be a non-strict Archimedean t-norm.

DEFINITION 1.2. (cf. Sklar [5]). A copula is a two-place function $C:I^2 \to I$ such that

(i) $C(1,x)=C(x,1)=x$, $C(0,x)=C(x,0)=0$ for all x in I;

(ii) $C(y_1,y_2)-C(y_1,x_2)-C(x_1,y_2)+C(x_1,x_2) \geq 0$ for all pairs (x_1,x_2), (y_1,y_2) in I^2 such that $x_1 \leq y_1$ and $x_2 \leq y_2$.

Any copula C is continuous and satisfies the inequalities:

$$W(x,y): = \max(x+y-1,0) \leq C(x,y) \leq \min(x,y) = :M(x,y). \tag{1.2}$$

Copulas play a fundamental role in probability theory since they are functions connecting joint distribution functions of random variables with their margins. Precisely, we quote here the following result due to A. Sklar.

THEOREM 1.2. Let X and Y be two positive random variables defined in a common probability space with continuous distribution functions F_X and F_Y, respectively, and with a continuous joint distribution $H_{X,Y}$. Then there exists a unique copula $C_{X,Y}$ such that

$$H_{X,Y}(u,v) = C_{XY}(F_X(u),F_Y(v))$$

for all $u,v \geq 0$.

Copulas which are simultaneously Archimedean t-norms are especially interesting. In particular, we have the following

THEOREM 1.3. (see Schweizer-Sklar [2]). Let T be an Archimedean t-norm additively generated by a function f, i.e.

$$T(x,y) = T(f)(x,y) = f^{(-1)}(f(x)+f(y)), \quad x,y \in I. \tag{1.3}$$

Then the following three conditions are pairwise equivalent:

(i) T is a copula;

(ii) f is convex;

(iii) T satisfies the Lipschitz condition:

$$T(z,y)-T(x,y) \leq z-x \quad \text{whenever} \quad x,y,x \in I \quad \text{and} \quad x \leq z.$$

2. STRICT T-NORMS BEING UNIFORMLY CLOSE COPULAS.

Let $G = T(g)$ be a strict t-norm with a convex generator g. Thus

$$G(x,y) = g^{-1}(g(x)+g(y)), \quad x,y \in I$$

and, by Theorem 1.3, G is a copula. First, we are going to show how to construct copulas $F = T(f)$ which are uniformly close to G, i.e. such that for a given $\varepsilon > 0$ one has

$$||T(f)-T(g)|| = ||F-G|| = \sup_{(x,y) \in I^2} |F(x,y)-G(x,y)| \leq \varepsilon.$$

THEOREM 2.1. Given an $\varepsilon \in (0,1)$ and a strictly decreasing convex function g mapping the unit interval I onto $[0,\infty]$, let ϕ be a concave bijection of $[0,\infty)$ onto itself satisfying the Ulam-Hyers inequality (see Hyers [10])

$$|\phi(x+y)-\phi(x)-\phi(y)| \leqq \eta := g(1-\varepsilon)$$

for all $x,y \in [0,\infty)$. Put

$$f(x) := \begin{cases} (\phi^{-1}\circ g)(x) & \text{for } x \in (0,1], \\ \infty & \text{for } x = 0. \end{cases}$$

Then $T(f)(x,y) := f^{-1}(f(x)+f(y))$, is a copula and

$$\|T(f) - T(g)\| \leqq \varepsilon .$$

PROOF. The convexity of g jointly with Theorem 1.3 imply that T(g) is a copula. Consequently, on account of (1.2),

$$g^{-1}(g(x)+g(y)) \geqq x+y-1$$

for all $x,y \in I$. Take arbitrary $u, v \in [0,\infty) \subset g(I)$; then $u=g(x)$ and $v=g(y)$ for some $x,y \in I$ whence

$$g^{-1}(u+v) \geqq g^{-1}(u)+g^{-1}(v) - 1$$

and therefore

$$\begin{aligned} |g^{-1}(u)-g^{-1}(v)| &= g^{-1}(\min(u,v))-g^{-1}(\max(u,v)) \\ &= g^{-1}(\min(u,v))-g^{-1}(\min(u,v)+|u-v|) \\ &\leqq g^{-1}(\min(u,v))-g^{-1}(\min(u,v))-g^{-1}(|u-v|)+1 \\ &= 1-g^{-1}(|u-v|). \end{aligned}$$

Now, for any $x,y \in (0,1]$, one has

$$\begin{aligned} |T(f)(x,y)-T(g)(x,y)| &= |f^{-1}(f(x)+f(y))-g^{-1}(g(x)+g(y))| \\ &= |(g^{-1}\circ\phi)(f(x)+f(y))-g^{-1}(g(x)+g(y))| \\ &\leqq 1-g^{-1}(|\phi(f(x)+f(y))-g(x)-g(y)|) \\ &= 1-g^{-1}(|\phi(f(x)+f(y))-\phi(f(x))-\phi(f(y))|) \\ &\leqq 1-g^{-1}(\eta) = \varepsilon , \end{aligned}$$

which proves the desired inequality because the expression just estimated vanishes whenever x=0 or y=0.

Observe that our bijection ϕ has to be increasing. In fact, ϕ being concave is continuous and therefore strictly monotonic; if ϕ were decreasing we would get

$$0 \leqq \phi(y) \leqq \phi(x) + \phi(y) - \phi(x+y) \leqq |\phi(x)+\phi(y)-\phi(x+y)| \leqq \eta$$

for all $x, y \in [0,\infty)$, whence we would deduce the boundedness of ϕ which is a contradiction. Therefore, ϕ^{-1} is increasing, too, and the concavity of ϕ implies the convexity of ϕ^{-1} and henceforth f. Thus T(f) is a copula and the proof is completed.

REMARK 2.1. Bijections ϕ spoken of in Theorem 1.3 actually exist. Indeed, take any positive real number c and any strictly increasing and concave mapping $\gamma:[0,\infty) \to [0,\frac{1}{3}\eta)$ with $\gamma(0) = 0$. Then the mapping $\phi:[0,\infty) \to [0,\infty)$ given by the formula

$$\phi(x) = cx + \gamma(x), \quad x \in [0,\infty),$$

satisfies all the conditions desired.

Another result in that spirit is the following

THEOREM 2.2. Given an $\varepsilon \in (0,1)$ let f and g be two convex and strictly decreasing functions from I onto $[0,\infty]$ such that

$$|f^{-1}(x)-g^{-1}(x)| \leqq \frac{1}{3}\varepsilon \tag{2.1}$$

for all $x \in [0,\infty)$. Then the operations $T(f)(x,y): = f^{-1}(f(x)+f(y))$ and $T(g)(x,y): = g^{-1}(g(x)+g(y))$ are two associative copulas such that

$$\|T(f) - T(g)\| \leqq \varepsilon.$$

PROOF. In order to show the last inequality fix arbitrarily a pair $(x,y) \in I^2$. We shall distinguish two cases.

1) $\max(x,y) \geqq 1-\varepsilon$. Since T(f) and T(g) are copulas we get from (1.2) that they are both minorized by W and majorized by M. Consequently,

$$|T(f)(x,y)-T(g)(x,y)| \leqq M(x,y)-W(x,y)=\min(x,y)-\max(x+y-1,0) \leqq 1-\max(x,y) \leqq \varepsilon.$$

2) $\max(x,y) < 1-\varepsilon$. Then $x+\varepsilon$ as well as $y+\varepsilon$ belongs to (0,1). Relation (2.1) implies that

$$g^{-1}(f(t)) \leqq t+\frac{1}{3}\varepsilon \text{ for all } t \in I .$$

Thus

$$g(x+\tfrac{1}{3}\varepsilon) \leqq f(x) \quad \text{and} \quad g(y+\tfrac{1}{3}\varepsilon) \leqq f(y) \tag{2.2}$$

and, subsequently, on account of (2.1), the monotonicity of g^{-1}, (2.2) and Theorem 1.3 (iii), we get

$$\begin{aligned}T(f)(x,y) &= f^{-1}(f(x)+f(y)) \leq g^{-1}(f(x)+f(y)) + \tfrac{1}{3}\varepsilon \\ &\leq g^{-1}(g(x+\tfrac{1}{3})+g(y+\tfrac{1}{3}))+\tfrac{1}{3}\varepsilon = T(g)(x+\tfrac{1}{3}\varepsilon,y+\tfrac{1}{3}\varepsilon)+\tfrac{1}{3}\varepsilon \\ &= (T(g)(x+\tfrac{1}{3}\varepsilon,y+\tfrac{1}{3}\varepsilon) - T(g)(x,y+\tfrac{1}{3}\varepsilon)) + (T(g)(x,y+\tfrac{1}{3}\varepsilon) \\ &- T(g)(x,y))+T(g)(x,y)+\tfrac{1}{3}\varepsilon \leq (x+\tfrac{1}{3}\varepsilon -x)+(y+\tfrac{1}{3}\varepsilon -y) \\ &+ T(g)(x,y)+\tfrac{1}{3}\varepsilon = T(g)(x,y)+\varepsilon .\end{aligned}$$

Interchanging the roles of f and g we obtain also that $T(g)(x,y) \leq T(f)(x,y) + \varepsilon$, which finishes the proof.

Theorem 2.1 and 2.2 can easily be applied to the study of nonparametric measures of dependence for random variables. Given two random variables X, Y in a common probability space and with a unique copula $C_{X,Y}$ we recall from [7] the forms of two well-known measures of independence:

$$\tau(X,Y) = 4 \sup \{|C_{X,Y}(u,v)-uv| \; : \; u,v \in I\}$$

$$\sigma(X,Y) = 12 \cdot \int_0^1 \int_0^1 |C_{XY}(u,v)-uv|\,du\,dv.$$

Then we have

COROLLARY 2.1. Given an $\varepsilon \in (0,1)$ let ϕ be a concave bijection of $[0,\infty)$ onto itself satisfying the Ulam-Hyers inequality

$$|\phi(x+y)-\phi(x)-\phi(y)| \leq -\ln(1-\varepsilon)$$

for all $x,y \in [0,\infty)$. Assume that X and Y are two random variables with

$$C_{XY}(u,v) := \exp[-\phi(\phi^{-1}(-\ln u) + \phi^{-1}(-\ln v))],$$

$u,v \in I$. Then

$$\tau(X,Y) \leq 4\varepsilon \quad \text{and} \quad \sigma(X,Y) \leq 12\varepsilon .$$

COROLLARY 2.2. Given an $\varepsilon \in (0,1)$ let f be a strictly decreasing convex function mapping the unit interval I onto $[0,\infty]$. Assume that X and Y are two random variables with $C_{XY} = T(f)$. If $|f^{-1}(x)-e^{-x}| \leq \varepsilon$ for all $x \in [0,\infty)$, then $\tau(X,Y) \leq 4\varepsilon$ and $\sigma(X,Y) \leq 12\varepsilon$.

3. NON-STRICT T-NORMS BEING UNIFORMLY CLOSE COPULAS.

Let F_a, $a > 0$, be the family of all strictly decreasing functions mapping the unit interval I onto the interval $[0,a]$. Obviously, F_a is a subfamily of the collection C(I) of all continuous real functions defined on I because the ranges of members from F_a are connected. In what follows, C(E) will always stand for the Banach space of all continuous real functions on a compact metric space E; C(E) is assumed to be endowed with the usual uniform convergence norm.

THEOREM 3.1. The transformation $T: F_a \to C(I^2)$ given by the formula

$$T(f)(x,y) := f^{(-1)}(f(x)+f(y)), \quad x,y \in I, \quad f \in F_a,$$

is continuous.

PROOF. Take any sequence $(f_n)_{n\in N}$ of elements from F_a uniformly convergent to an $f \in F_a$ and fix arbitrarily an $\varepsilon > 0$. Since f^{-1} is uniformly continuous one may find a $\delta > 0$ such that for all $s,t \in [0,a]$

$$|f^{-1}(s)-f^{-1}(t)| < \tfrac{1}{2}\varepsilon \quad \text{provided that } |s-t| < \delta . \tag{3.1}$$

As f is the uniform limit of $(f_n)_{n\in N}$, there exists an $n_o \in N$ such that for all $n \geq n_o$ and $x \in I$ one has $|f_n(x)-f(x)| < \delta$. Consequently, taking $s: = f(x)$ and $t: = f_n(x)$ in (3.1) we get $|x-f^{-1}(f_n(x))| \leq \frac{1}{2}\varepsilon$ for all $n \geq n_o$ and $x \in I$; hence

$$|f_n^{-1}(z)-f^{-1}(z)| < \tfrac{1}{2}\varepsilon \quad \text{for all } n \geq n_o \text{ and } z \in [0,a]. \tag{3.2}$$

On the other hand, putting

$$m_a(g)(x,y): = \min(g(x)+g(y),a),\ x,y \in I,\ \ g \in F_a,$$

one may easily check that the sequence $(m_a(f_n))_{n\in N}$ tends uniformly to $m_a(f)$ on the unit square. Therefore, there exists an $n_1 \in N$ such that

$$n \geq n_1 \ \text{ implies } \ \|m_a(f_n)-m_a(f)\| < \delta \ . \tag{3.3}$$

Finally, for any pair $(x,y) \in I^2$ one has (see (1.1))

$$\begin{aligned}
|T(f_n)(x,y)-T(f)(x,y)| &= |f_n^{(-1)}(f_n(x)+f_n(y))-f^{(-1)}(f(x)+f(y))| \\
&= |f_n^{-1}(m_a(f_n)(x,y))-f^{-1}(m_a(f)(x,y))| \\
&\leq |f_n^{-1}(m_a(f_n)(x,y))-f^{-1}(m_a(f_n)(x,y))| \\
&\quad + |f^{-1}(m_a(f_n)(x,y)-f^{-1}(m_a(f)(x,y))| \\
&< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon
\end{aligned}$$

whenever $n \in N$, $n \geq \max(n_0,n_1)$, by means of the subsequent use of (3.2), (3.3) and (3.1). This shows that

$$\|T(f_n)-T(f)\| \to 0 \quad \text{as} \quad n\to\infty$$

and finishes the proof.

As an immediate consequence of this result we obtain the following

THEOREM 3.2. Given an $\varepsilon > 0$ let g be a strictly decreasing and convex function mapping I onto $[0,a]$. Then there exists a $\delta > 0$ such that for all strictly decreasing and convex surjections $f: I \to [0,a]$ fulfilling the condition

$$\|f-g\| < \delta$$

the associative copulas

$$T(f)(x,y)=f^{(-1)}(f(x)+f(y)),\ T(g)(x,y)=g^{(-1)}(g(x)+g(y)),\ x,y \in I,$$

are ε-uniformly close, i.e.

$$\|T(f)-T(g)\| < \varepsilon .$$

Observe, however, that Theorem 3.2 has an existence character ("there exists a $\delta > 0$"). The following result shows how to produce associative copulas T(f) ε-close to T(g) but requires stronger assumptions on the given generator g and refers only to some specific generators f.

THEOREM 3.3. Given an $\varepsilon > 0$ let g be a strictly decreasing and convex function from I onto [0,a] such that

$$m: = \inf \left\{ \left| \frac{g(x)-g(y)}{x-y} \right| \quad : \quad x,y \in I \ , \quad x \neq y \ \right\} > 0. \tag{3.4}$$

Assume ϕ to be any concave bijection of [0,a] onto itself such that

$$|\phi(x)-x| \leq \frac{1}{3} m \, \varepsilon \ , \quad x \in [0,a] \ , \tag{3.5}$$

and put $f: = \phi^{-1} \circ g$. Then the operations

$$T(g)(x,y): = g^{(-1)}(g(x)+g(y)) \text{ and } T(f)(x,y): = f^{(-1)}(f(x)+f(y)),$$

represent ε-uniformly close associative copulas, i.e.

$$\|T(f)-T(g)\| \leq \varepsilon \ . \tag{3.6}$$

PROOF. Obviously, relation (3.4) implies that

$$|g^{-1}(x)-g^{-1}(y)| \leq \frac{1}{m} |x-y| \ , \qquad x,y \in [0,a]. \tag{3.7}$$

On the other hand, assumption (3.5) leads immediately to the Ulam-Hyers inequality

$$|\phi(x+y)-\phi(x)-\phi(y)| \leq m \cdot \varepsilon \tag{3.8}$$

for all $x,y \in [0,a]$ such that $x+y$ is in [0,a], which, in particular, forces ϕ to be strictly increasing (cf. the proof of Theorem 2.1) and hence T(f) to be a copula.

To prove (3.6), fix a pair $(x,y) \in I^2$ and consider the following four cases:

(a) $T(f)(x,y)=T(g)(x,y)=0$;

(b) both $T(f)(x,y)$ and $T(g)(x,y)$ are positive;

(c) $T(g)(x,y) = 0 < T(f)(x,y)$;

(d) $T(f)(x,y) = 0 < T(g)(x,y)$.

We have to show that

$$\gamma(x,y): = |T(f)(x,y)-T(g)(x,y)| \leq \varepsilon$$

which becomes trivial in case (a). Assuming (b) one gets

$$\begin{aligned} \gamma(x,y) &= |f^{-1}(f(x)+f(y))-g^{-1}(g(x)+g(y))| \\ &= |g^{-1}(\phi(f(x)+f(y)))-g^{-1}(g(x)+g(y))| \\ &\leq \frac{1}{m} |\phi(f(x)+f(y))-\phi(f(x))-\phi(f(y))| \leq \varepsilon \end{aligned}$$

by means of the definition of f, (3.7) and (3.8).

In case (c) one has $g(x)+g(y) \geq a > f(x)+f(y)$ whence

$$\begin{aligned}\gamma(x,y) &= f^{-1}(f(x)+f(y))-g^{-1}(\phi(f(x)+f(y)))-g^{-1}(a)\\ &\leq \frac{1}{m}(a-\phi(f(x)+f(y))) \leq \frac{1}{m}(g(x)+g(y)-\phi(f(x)+f(y)))\\ &= \frac{1}{m}(\phi(f(x))+\phi(f(y))-\phi(f(x)+f(y))) \leq \varepsilon ,\end{aligned}$$

on account of (3.7) and (3.8), again.

Finally, if (d) occurs then $g(x)+g(y) < a \leq f(x)+f(y)$ whence

$$\begin{aligned}\gamma(x,y) &= g^{-1}(g(x)+g(y))-g^{-1}(a) \leq \frac{1}{m}(a-g(x)-g(y))\\ &\leq \frac{1}{m}((f(x)-g(x))+(f(y)-g(y))) \leq \frac{1}{m}\cdot 2\frac{1}{3}m - \frac{2}{3}\varepsilon < \varepsilon ,\end{aligned}$$

because of (3.7) and the relation

$$|f(t)-g(t)| \leq \frac{1}{3}m\varepsilon , \quad t \in I ,$$

resulting directly from (3.5) and the definition of f. This completes the proof.

REMARK 3.1. Any continuously differentiable and convex surjection $g: I \to [0,a]$ such that $g'(1) < 0$ satisfies all the requirements concerning the function g occuring in Theorem 3.3. In fact, convexity implies that g' is increasing whence $g'(x) \leq g'(1) < 0$, $x \in I$, i.e. g is strictly decreasing. On the other hand, for any $x,y \in I$, $x \neq y$, one has

$$\left|\frac{g(x)-g(y)}{x-y}\right| = |g'(\lambda)| = -g'(\lambda) \geq g'(1) =: m_o > 0$$

for some λ between x and y; therefore $\inf_{\substack{x,y\in I\\ x\neq y}} \frac{|g(x)-g(y)|}{x-y} \geq m_o > 0$, as desired.

4. REVERSE IMPLICATIONS.

Having two uniformly close Archimedean t-norms F and G, a natural question arises whether they admit uniformly close generators. In other words, if $F=T(f)$, $G=T(g)$ and

$$|g^{(-1)}(g(x)+g(y))-f^{(-1)}(f(x)+f(y))| \leq \varepsilon, \quad x,y \in I , \tag{4.1}$$

we ask whether these exist two positive constants α and β such that

$$|\alpha f(x)-\beta g(x)| \leq \varepsilon , \quad x \in I . \tag{4.2}$$

The answer, in affirmative, is trivial in the case where both T(f) and T(g) are non-strict, say, f maps I onto [0,a] and $g(I) = [0,b]$ for some $a,b \in (0,\infty)$. Then, taking $\alpha=\beta: = \frac{\varepsilon}{a+b}$ we get

$$\|\alpha f-\alpha g\| \leq \alpha\cdot\sup_{t\in I} f(t) + \alpha\sup_{t\in I} g(t) = \alpha(a+b) = \varepsilon$$

and obviously, $T(\alpha f)=T(f)$ as well as $T(\alpha g)=T(g)$.

If one of the generators f and g is bounded whereas the other is not, then automatically relation (4.2) fails to hold for any positive α and β. Therefore, it remains to consider the case where both f and g are unbounded. Observe that $\varepsilon < 1$ are the only interesting constants in (4.1) since, plainly, for any two strictly decreasing functions f and g mapping the unit interval I onto $[0,\infty]$ one has

$$|g^{-1}(t)-f^{-1}(s)| \leq 1 \tag{4.3}$$

for all s, $t \varepsilon [0,\infty]$. Nevertheless even very regular ε-uniformly close strict t-norms (copulas), $\varepsilon < 1$, may fail to have close generators. To visualize this, take

$$f(x):=\begin{cases}\frac{1}{x}-1 & \text{for } x(0,1]\\ \infty & \text{for } x=0\end{cases} \qquad \text{and} \quad g(x):=\begin{cases}-\ln x & \text{for } x\in(0,1]\\ \infty & \text{for } x=0;\end{cases}$$

then

$$T(f)(x,y) = f^{-1}(f(x)+f(y)) = \begin{cases}\frac{xy}{x+y-xy} & \text{for } (x,y)\in[0,1]^2\setminus\{(0,0)\}\\ 0 & \text{for } (x,y)=(0,0)\end{cases}$$

and

$$T(g)(x,y) = g^{-1}(g(x)+g(y)) = xy.$$

The continuous function

$$\gamma(x,y) := T(f)(x,y) - T(g)(x,y),$$

vanishes on the boundary of the unit square; therefore, the value $\|\gamma\| = \max_{(x,y)\in I^2} |\gamma(x,y)|$ is attained at an interior point of the unit square. Since $z:=(\frac{3-\sqrt{5}}{2},\frac{3-\sqrt{5}}{2})$ is the only critical point of γ in $(0,1)^2$ and $\gamma(z)=\frac{1}{2}(5\sqrt{5}-11) > 0$ we have

$$0 \leq \gamma(x,y) = \frac{xy}{x+y-xy} - xy \leq \varepsilon_o := \tfrac{1}{2}(5\sqrt{5}-11) < 0.0903. \tag{4.4}$$

Thus, the copulas T(f) and T(g) are ε_o-uniformly close, but for any positive numbers α and β the difference

$$\alpha f(x) - \beta g(x) = \frac{\alpha}{x} + \beta \ln x - \alpha$$

tends to infinity as x approaches zero from the right.

What about the distance between f^{-1} and g^{-1}? In the light of (4.3) we always have

$$d(x) := |f^{-1}(x) - g^{-1}(x)| \leq 1 \quad \text{for} \quad x \in [0,\infty]. \tag{4.5}$$

On the other hand, since $f^{-1}(0) = g^{-1}(0) = 1$ and $\lim_{x\to\infty} f^{-1}(x) = \lim_{x\to\infty} g^{-1}(x) = 0$, the continuity of f^{-1} and g^{-1} forces d to be upper-bounded by an a priori given positive constant except for a compact subinterval of $(0,\infty)$. One may expect however, that having (4.1) we get $d(x) \leqq \varepsilon$ for $x \geqq 0$. This is not true: taking f and g as above (see (4.4))

$$|f^{-1}(f(x)+f(y)) - g^{-1}(g(x)+g(y))| \leqq 0.0903, \quad x,y \in I,$$

and a standard calculation proves that

$$d(x) = |f^{-1}(x)-g^{-1}(x)| = \left|\frac{1}{1+x} - e^{-x}\right| > \frac{1}{5} \quad \text{for an } x \in (2.6, 2.7).$$

Nevertheless, one may show that

$$d(x) < 0.20364 \quad \text{for all} \quad x \in [0,\infty],$$

which is definitely more interesting than (4.5).

In the general case we were able to state only the following

PROPOSITION 4.1. For any two strictly decreasing convex and unbounded generators $f,g\colon I \to [0,\infty]$ such that $||T(f)-T(g)|| \leqq \varepsilon < 1$ there exists a positive $\delta < 1$ such that $|f^{-1}(x)-g^{-1}(x)| \leqq \delta$ for all $x \in [0,\infty]$.

PROOF. As we have remarked before, the distance function $d := |f^{-1}-g^{-1}|$ does not exceed ε outside a compact interval $[\alpha,\beta] \subset (0,\infty)$. Since f and g and hence also f^{-1} and g^{-1} are necessarily continuous so is the distance function d and it suffices to take

$$\delta := \max(\varepsilon, \max_{x\in[\alpha,\beta]} d(x))$$

which, plainly, is strictly less than one.

This, however, is by no means satisfactory because the important question whether the (not necessarily linear) function $\varepsilon \to \delta(\varepsilon)$ tends to zero as $\varepsilon \to 0$, remains unanswered.

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