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NOTE ON LEGENDRE NUMBERS

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ABSTRACT. The definition and basic properties of Legendre Numbers are reviewed here. We then develop some new properties and identities involving sums of Legendre Numbers, including clarification of a statement in the paper of Haggard [1].

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INTRODUCTION

Recently P. W. Haggard [1] introduced Legendre Numbers, discussed various of their properties, and evaluated certain related infinite series and integrals. In this note we review some of these ideas and discuss some further results.

1. LEGENDRE NUMBERS.

The Legendre polynomials $P_n(x)$ are defined [2] by the generating function

$$(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n \tag{1.1}$$

and the Associated Legendre functions are defined by

$$P_n^m(x) = (1 - x^2)^{\frac{m}{2}} D^m(P_n(x)). \tag{1.2}$$

Recently P. W. Haggard [1] defined the Legendre Number, P_n^m, to be $P_n^m(0)$ and studied some of their basic properties. By the well-known Rodrigue's formula [2]

$$P_n(x) = \frac{1}{2^n \cdot n!} D^n((x^2 - 1)^n), \tag{1.3}$$

we see that

$$P_n^m(x) = \frac{(1 - x^2)^{\frac{m}{2}}}{2^n \cdot n!} D^{m+n}((x^2 - 1)^n) \tag{1.4}$$

and consequently

$$P_n^m = P_n^{(m)}(0) \ , \tag{1.5}$$

where $P_n^{(m)}(0)$ is the value of the mth derivative of $P_n(x)$ at $x = 0$. Haggard [1] deduced the following explicit formula from (1.4).

$$P_n^m = \begin{cases} 0 , & m+n \text{ odd} \\ 0 , & m > n \text{ and} \\ \dfrac{(-1)^{\frac{n-m}{2}}(n+m)!}{2^n(\frac{n+m}{2})!(\frac{n-m}{2})!} , & m+n \text{ even}, m \le n. \end{cases} \tag{1.6}$$

He also gave a table of P_n^m for $0 \le m, n \le 8$.

We note that (1.6) follows directly from (1.1). In fact by (1.1) and the Binomial theorem

$$\sum_{n=0}^{\infty} P_n(x)t^n = \{1 - t(2x-t)\}^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n \cdot n!} t^n(2x-t)^n$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (2x)^{n-k} t^{n+k}$$

$$= 1 + \sum_{m=1}^{\infty} t^m \sum_{\frac{m}{2} \le n \le m} \frac{(-1)^{m-n}(2n)!}{2} \binom{n}{m-n} x^{2n-m}$$

so that

$$P_n(x) = 2^{-n} \sum_{i=0}^{n/2} (-1)^i \binom{n-i}{i} \binom{2(n-i)}{n-i} x^{n-2i}$$

Writing m for n - 2i in this, we get

$$P_n(x) = 2^{-n} \sum_{\substack{m=0 \\ m+n \text{ even}}}^{n} (-1)^{\frac{n-m}{2}} \binom{(n+m)/2}{(n-m)/2} \binom{n+m}{(n+m)/2} x^m .$$

Now since $P_n(x) = \sum_{m=0}^{n} \frac{P_n^{(m)}(0)}{m!} x^m$, we get (1.6).

2. INTEGER VALUES OF P_n^m

In this section we prove that for $P_n^m \neq 0$, then P_n^m is an integer iff m = n. For this, let [x] denote the largest integer $\le x$, and for prime p and $n \ge 1$, let H(p,n) denote the highest power of p dividing n. Then it is well known, due to Legendre (cf. [3], p. 67), that

$$H(p,n!) = \sum_{r=1}^{\infty} \left[\frac{n}{p^r}\right] , \tag{2.1}$$

THEOREM 2.1 The highest power of 2 dividing the denominator of P_n^{n-2k}, $n \ge 1$ and $k \ge 0$, when expressed in its lowest terms, is k + H(2,k!). In particular, a non-zero Legendre number P_n^m is an integer iff m = n.

PROOF. By 1.6 and 2.1, letting m = n - 2k, we see that the highest power of 2 dividing the denominator of P_n^{n-2k} (in its reduced form) is given by

$$H(2,2^n(\tfrac{n+n-2k}{2})!(\tfrac{n-n+2k}{2})!)$$

$$= n + \sum_{r=1}^{\infty}\left[\frac{n+(n-2k)}{2^r+1}\right] + \sum_{r=1}^{\infty}\left[\frac{n-(n-2k)}{2^{r+1}}\right] - \sum_{r=1}^{\infty}\left[\frac{n+(n-2k)}{2^r}\right]$$

$$= n + \sum_{r=1}^{\infty}\left[\frac{n-k}{2^r}\right] - \sum_{r=0}^{\infty}\left[\frac{n-k}{2^r}\right] + \sum_{r=1}^{\infty}\left[\frac{k}{2^r}\right]$$

$$= n - (n-k) + \sum_{r=1}^{\infty}\left[\frac{k}{2^r}\right]$$

$$= k + H(2,k!).$$

Hence, if $m = n$, then $k = 0$, and the highest power of 2 dividing the denominator of P_n^n is zero and P_n^n is an integer. If $k = 0$, then $m = n$.

3. SUMS INVOLVING P_n^m.

Haggard [1] proved that

$$\sum_{n=0}^{\infty} P_n^0 = 2^{-1/2}, \tag{3.1}$$

and for $k \geq 1$

$$\sum_{n=k}^{\infty} P_n^k = 1\cdot 3\cdot 5\cdot 7 \text{----} (2k-1)\, 2^{-(\frac{2k+1}{2})}. \tag{3.2}$$

However, we note that his arguments prove only that the stated sums in (3.1) and (3.2) are in the sense of Abel. In fact, as we show later, the series

$$\sum_{n=k}^{\infty} P_n^k, \text{ for fixed } k \geq 0, \text{ converges iff } k = 0.$$

To see this, using Stirling's formula, viz.

$$n! \sim \sqrt{2\pi}\, n^{n+1/2} e^{-n}, \text{ as } n \to \infty$$

we have

$$\binom{2n}{n} \sim 2^{2n}(n\pi)^{-1/2},$$

and hence

$$P_{2n}^0 = (-1)^n \binom{2n}{n} 2^{-2n} \sim \frac{(-1)^n}{\sqrt{n\pi}}$$

Also, since the sequence $\{|P_{2n}^0|\}_n = 0,1,2,\ldots$ is decreasing, the series $\sum_{n=0}^{\infty} P_{2n}^0$ converges.

Now let $k \geq 1$ be fixed. Again by Stirling's formula, we see that

$$P_{2n+k}^k = (-1)^n \frac{(2n+2k)!}{n!(n+k)!2^{2n+k}} \sim (-1)^n n^{k-1/2} 2^k \pi^{-1/2}$$

and hence the series $\sum_{n=0}^{\infty} P_{2n+k}^k$, for fixed $k \geq 1$, actually diverges.

However, some interesting sums involving reciprocals of Legendre numbers yield the following results.

THEOREM 3.1 For $|x| < 1$

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{n P_{2n}^{0}} = \frac{2x \operatorname{Sin}^{-1} x}{\sqrt{1 - x^2}} \tag{3.3}$$

PROOF. It is known from Lehmer [4] that for $|x| < 1$

$$\sum_{n=1}^{\infty} \frac{(2x)^{2n}}{n \binom{2n}{n}} = \frac{2x \operatorname{Sin}^{-1} x}{\sqrt{1 - x^2}}$$

and (3.3) is a reformulation of this.

COROLLARY 3.1 For $|x| < 1$

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{n^2 P_{2n}^{0}} = 2(\operatorname{Sin}^{-1} x)^2 \tag{3.4}$$

by dividing (3.3) by x and integrating both sides,

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{P_{2n}^{0}} = \frac{x^2}{1 - x^2} + \frac{x \operatorname{Sin}^{-1} x}{(1 - x^2)^{3/2}} \tag{3.5}$$

by differentiation of (3.3) and multiplying by x, and then,

$$\sum_{n=1}^{\infty} \frac{x^{2n}}{n P_{2n}^{0}} = -\frac{2x(\operatorname{Sinh}^{-1} x)}{\sqrt{1 + x^2}} \tag{3.3'}$$

$$\sum_{n=1}^{\infty} \frac{x^{2n}}{n^2 P_{2n}^{0}} = -2(\operatorname{Sinh}^{-1} x)^2 \tag{3.4'}$$

$$\sum_{n=1}^{\infty} \frac{x^{2n}}{P_{2n}^{0}} = -\left\{\frac{x^2}{1 + x^2} + \frac{x \operatorname{Sinh}^{-1} x}{(1 + x^2)^{3/2}}\right\} \tag{3.5'}$$

by replacing x by ix in 3.3, 3.4 and 3.5.

REMARK 3.1 Since $P_{2n-1}^{1} = -2nP_{2n}^{0}$, results corresponding to Theorem 3.1 and Corollary 3.1 can be formulated for sums involving P_{2n-1}^{1}. For various special cases we refer the reader to the very interesting paper of D. H. Lehmer [4]. However, it appears that obtaining a closed expression for the sums of series such as

$$\sum_{\substack{n=m \\ n+m \text{ even}}}^{\infty} \frac{x^{2n}}{P_n^m}$$

, for larger m, is a difficult problem.

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