

AN APPLICATION OF STRESS ENERGY TENSOR TO THE VANISHING THEOREM OF DIFFERENTIAL FORMS

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ABSTRACT. The author applies the stress energy of differential forms to study the vanishing theorems of the Liouville type. It is shown that for a large class of underlying manifolds such as the Euclidean n -space, the complex n -space, and the complex hyperbolic space form, if any vector bundle valued p -form with conservative stress energy tensor is of finite norm or slowly divergent norm, then the p -form vanishes. This generalizes the recent results due to Hu and Sealey.

KEYWORDS AND PHRASES. Conservation, stress energy tensor, complex hyperbolic space form, vector bundle valued p -form.

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1. INTRODUCTION.

The classical Liouville theorem states that any bounded harmonic function defined on the entire plane is a constant function. Later many authors generalized the theorem. Yau [1] generalized it to harmonic functions on complete Riemannian manifolds with non-negative Ricci curvature. It was further extended to harmonic maps between Riemannian manifolds by Eells and Sampson [2], Cheng [3], Garbar et al. [4], Hu [5], Xin [6], Sealey [7] and others.

It is well known that if the field equation of a physical system is obtained by means of calculating the derivative of the action functional I , we can define a so called stress energy tensor S such that S is conservative at the critical values of I . In particular, the stress at a point of an elastic body is described by a tensor (S_{ij}) in the Euclidean space R^3 . The law of conservation of momentum about some origin implies the stress energy tensor S_{ij} be symmetric and the system be in equilibrium means that S_{ij} be divergence free. Recently, Baird and Eells [1] proved that if the map between Riemannian manifolds is harmonic, then its stress energy tensor is conservative. This might make the harmonic map a new physical explanation.

The purpose of the paper is to apply the stress energy tensor of p-form with values in a vector bundle over some kind of Riemannian manifolds to the vanishing theorem of the Liouville type. The main result is the following.

THEOREM 1.1. Let U^n be the complex hyperbolic space form and F a Hermitian vector bundle over U^n . Let ω be a p-form with values in F , $p < n$. If ω is of finite or slowly divergent norm over U^n and its stress energy tensor is conservative, then $\omega = 0$.

It must be pointed out that this is a more general vanishing theorem of vector bundle valued p-form. We all know that there are many important maps between Riemannian manifolds such as harmonic, relative harmonic [5], holomorphic [8] and relative affine maps [9] whose differentials possess the free divergent stress energy tensors. Sealey [7] proved that if the norm of a harmonic 1-form with values in a vector bundle over U^n is integrable, then the 1-form vanishes. Therefore, Theorem 1 means that under a weaker condition than Sealey's, a stronger conclusion still holds.

In section 2 after introducing some notations, we establish an integral formula of the stress energy tensor and some kind of radial variational formula for p-forms by using the Lie derivative. Besides being applied to the proof of Theorem 1, these formulas are of interest in its own right. In section 3, we prove Theorem 1.1. by means of estimating the first variation of the norm of p-form motivated by the method in [4]. Finally, we expand our result to the general case where the underlying manifold can be the complex n-space or some kind of conformal flat Riemannian manifolds.

2. NOTATIONS AND LEMMAS.

For simplicity, we only state some notations and formulas for the Riemannian vector bundle over real Riemannian manifold, those of Hermitian vector bundle over Kaehler manifold can be written correspondently. Let F be a Riemannian vector bundle over Riemannian manifold M . We define an inner product in $\wedge^p T_x^* M \otimes F_x$ by setting

$$\langle \omega, \theta \rangle = \sum_{i_1 < \dots < i_p} \langle \omega(e_{i_1}, \dots, e_{i_p}), \theta(e_{i_1}, \dots, e_{i_p}) \rangle \quad (2.1)$$

where $\omega, \theta \in \wedge^p T_x^* M \otimes F_x$, $\{e_1, \dots, e_n\}$ is any orthonormal basis of $T_x M$. Integrating this pointwise inner product over M gives an inner product in $A^p(F)$, which denotes the space of sections of $\wedge^p T^* M \otimes F$. For a linear connection ∇ on F we define the covariant derivative by

$$(\nabla_Y \omega)(X_1, \dots, X_p) = \nabla_Y \omega(X_1, \dots, X_p) - \sum_{i=1}^p \omega(X_1, \dots, \nabla_Y X_i, \dots, X_p) \quad (2.2)$$

for $Y, X \in A(TM)$. The exterior differential $d: A^p(F) \rightarrow A^{p+1}(F)$ is given by

$$d\omega(X_1, \dots, X_{p+1})$$

$$\begin{aligned}
&= \sum_{i=1}^{p+1} (-1)^{i+1} \nabla_{X_i} \omega(X_1, \dots, \hat{X}_i, \dots, X_{p+1}) \\
&+ \sum_{i < j} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{p+1})
\end{aligned} \tag{2.3}$$

where $X_i \in A(TM)$, $1 \leq i, j \leq p+1$, \hat{X}_i means X_i is deleted. The codifferential $\delta: A^p(F) \rightarrow A^{p-1}(F)$, $p > 1$, is given by

$$\delta(X_1, \dots, X_{p-1}) = - \sum_{i=1}^n \nabla_{e_i} \omega(e_i, X_1, \dots, X_{p-1}). \tag{2.4}$$

The generalized Laplace-Beltrami operator for p -form with values in F is defined by $\Delta = d\delta + \delta d$. Any p -form ω with values in F satisfying the equations $d\omega = 0$, $\delta\omega = 0$ is called harmonic. If M is a manifold without boundary and ω is of compact support, then ω is harmonic iff $\nabla\omega = 0$.

The symmetric square $\omega * \omega$ of $\omega \in A^p(F)$ is a symmetric F -valued 2-tensor defined by

$$\omega * \omega(X, Y) = \langle i_X \omega, i_Y \omega \rangle \tag{2.5}$$

where $X, Y \in A(TM)$, i_X denotes the contraction with X . We call

$$E_\omega = \int_M e_\omega * 1 \tag{2.6}$$

the norm of ω over M where $e_\omega = |\omega|^2/2$. The stress energy tensor of ω is given by

$$S_\omega = e_\omega g - \omega * \omega \tag{2.7}$$

in which g is the metric tensor of M .

LEMMA 2.1. Suppose X be a vector field with compact support over M and ω p -form with values in a vector bundle, then

$$\int_M \langle L_X \omega, \omega \rangle * 1 = \int_M \operatorname{div} S_\omega(X) * 1 + \int_{\partial M} \omega * \omega(X, \partial/\partial n) * 1 \tag{2.8}$$

in which $\operatorname{div} S_\omega$ is the divergence of S_ω and $\partial/\partial n$ the unit outward normal to ∂M .

PROOF. Since [6]

$$\operatorname{div} S_\omega(X) = \langle i_X d\omega, \omega \rangle + \langle \delta\omega, i_X \omega \rangle, \tag{2.9}$$

we get

$$\begin{aligned}
\langle L_X \omega, \omega \rangle &= \langle i_X d\omega + di_X \omega, \omega \rangle \\
&= \operatorname{div} S_\omega(X) + \langle di_X \omega, \omega \rangle - \langle \delta\omega, i_X \omega \rangle
\end{aligned} \tag{2.10}$$

Integrating and using Green theorem immediately complete the proof.

Let us consider a special case of the above formula. Assume $\phi: M \rightarrow N$ be a smooth map from a Riemannian manifold M without boundary to Riemannian manifold N . According to Lemma 1, we know that

$$\int_M \langle L_x d\phi, d\phi \rangle *1 = \int_M \langle d\phi(X), \delta d\phi \rangle *1 \tag{2.11}$$

so

$$\text{div } S_\phi = \langle \tau_\phi, d\phi \rangle \tag{2.12}$$

where S_ϕ is the stress energy tensor of ϕ and $\tau_\phi = \delta d\phi$ is the torsion field of ϕ . Thus, we easily prove

COROLLARY (Baird and Eells [10]). The stress energy tensor of a harmonic map from M to N must be conservative. Conversely, if ϕ is a differential submersion and its differential $d\phi$ is of conservative stress energy tensor, then ϕ is harmonic.

LEMMA 2.2. Let $U^n(\rho)$ be an open disc with radius ρ in the complex n -space C^n and ω be an F -valued p -form over $U^n(\rho)$. Suppose ω_t be an l -parameter family of p -forms such that

$$\omega_t(z_1, \dots, z_p) = t^p (\phi_t^* \omega_{tz})(z_1, \dots, z_p) \tag{2.13}$$

where $z \in U^n(\rho)$, $0 < t \leq \rho$, $\{z_1, \dots, z_p\}$ are the constant vector fields over $U^n(\rho)$, and ϕ_t^* is induced from parallel transport in F along the straight line from tz to z . Then we have

$$\frac{d}{dt} \omega_t \Big|_{t=1} = L_{r\partial/\partial r} \omega. \tag{2.14}$$

PROOF. This depends largely on a direct computation. One can easily see that at $t = 1$

$$\frac{d}{dt} (\phi_t^* \omega_{tz})(z_1, \dots, z_p) = \nabla_{r\partial/\partial r} \omega(z_1, \dots, z_p) \tag{2.15}$$

Therefore at $t=1$

$$\frac{d}{dt} \omega_t(z_1, \dots, z_p) = \nabla_{r\partial/\partial r} \omega(z_1, \dots, z_p) + p\omega(z_1, \dots, z_p) \tag{2.16}$$

On the other hand, we may assume $z_i = \partial/\partial z_i$ or $\partial/\partial \bar{z}_i$. Thus, $[z_i, z_j] = 0$ and $[r\partial/\partial r, z_i] = -z_i$. It follows that

$$\begin{aligned} (L_{r\partial/\partial r} \omega)(z_1, \dots, z_p) &= (i_{r\partial/\partial r} d\omega + di_{r\partial/\partial r} \omega)(z_1, \dots, z_p) \\ &= d\omega(r\partial/\partial r, z_1, \dots, z_p) + \sum_{i=1}^p (-1)^{i+1} \nabla_{z_i} \omega(r\partial/\partial r, z_1, \dots, \hat{z}_i, \dots, z_p) \\ &= \nabla_{r\partial/\partial r} \omega(z_1, \dots, z_p) + p\omega(z_1, \dots, z_p). \end{aligned} \tag{2.17}$$

The proof is completed.

From lemmas 2.1, 2.2 we immediately obtain

LEMMA 2.3. If ω is an F-valued p-form with conservative stress energy tensor, then $d\|\omega_t\|_\rho^2 / dt > 0$ at $t = 1$.

Finally, we give the following.

DEFINITION. Let F be a Hermitian vector bundle over U^n and ω be a p-form with values in F. The norm of ω over U^n called slowly divergent if

$$\int_{U^n} e_\omega *1 = \infty \quad \text{and} \quad \int_0^1 \frac{f(r)}{1-r} dr = \infty$$

hold for any positive continuous function $f(r)$ in $(0,1)$ (for example $f(r) = 1 / \log(1-r)$), we have

$$\int_{U^n} f(r)e_\omega *1 < \infty.$$

3. THE PROOF OF THEOREM 1.1.

After establishing the preceding lemmas, it is not difficult to prove Theorem 1. We represent the complex hyperbolic space form by the open unit disc U^n in C^n with the complete Kaehler metric [11]

$$g = 4 \frac{(1 - \sum z_\alpha \bar{z}_\alpha) (\sum dz_\alpha dz_\alpha^-) + (\sum \bar{z}_\alpha dz_\alpha) (\sum z_\alpha d\bar{z}_\alpha^-)}{(1 - \sum z_\alpha \bar{z}_\alpha)^2} \tag{3.1}$$

It is obvious to see that $T_z^{1,0}(U^n)$ and $T_z^{0,1}(U^n)$ have respectively the orthonormal bases

$$\left\{ \frac{1-r^2}{2} \frac{\partial}{\partial z_1}, \frac{\sqrt{1-r^2}}{2} \frac{\partial}{\partial z_2}, \dots, \frac{\sqrt{1-r^2}}{2} \frac{\partial}{\partial z_n} \right\}$$

and

$$\left\{ \frac{1-r^2}{2} \frac{\partial}{\partial \bar{z}_1}, \frac{\sqrt{1-r^2}}{2} \frac{\partial}{\partial \bar{z}_2}, \dots, \frac{\sqrt{1-r^2}}{2} \frac{\partial}{\partial \bar{z}_n} \right\}$$

For convenience, we may assume that ω is of degree (k,h) , $k+h = p$. Then ω can be expressed as

$$\omega = \sum f_{i_1 \dots i_k j_1 \dots j_h} dz_{i_1} \wedge \dots \wedge dz_{i_k} \wedge d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_h}. \tag{3.2}$$

To begin with we calculate the inner product $|\omega_t|^2$ at the point $(z_1, 0, \dots, 0)$. At that time, the Kaehler metric is reduced to

$$g = \frac{4}{(1-r^2)^2} (dz_1 d\bar{z}_1 + (1-z^2) \sum_2^n dz_\alpha d\bar{z}_\alpha) \tag{3.3}$$

We can decompose ω_t to $\omega_t = \omega_t^1 + \omega_t^2 + \omega_t^3$ among which ω_t^1 consists of such p-forms that do not contain the factors dz_1 and $d\bar{z}_1$ those of ω_t^2 contain factor dz_1 or $d\bar{z}_1$ and ω_t^3 are the remains. Using (2.13), (3.3), we can compute that

$$\begin{aligned} |\omega_t^1|^2 &= t^{2p} |\omega^1|_{tz}^2 \frac{(1-r^2)^p}{(1-t^2 r^2)^p}, \\ |\omega_t^2|^2 &= t^{2p} |\omega^2|_{tz}^2 \frac{(1-r^2)^{p+1}}{(1-t^2 r^2)^{p+1}}, \\ |\omega_t^3|^2 &= t^{2p} |\omega^3|_{tz}^2 \frac{(1-r^2)^{p+2}}{(1-t^2 r^2)^{p+2}}, \end{aligned} \tag{3.4}$$

Then, since U^n is a homogeneous space, the restriction on z can be removed and these expressions hold for all $z \in U^n$. Denote dV and dS be the volume elements of U^n and the hypersphere S_r in U^n respectively. We have $dV = 2drdS / (1-r^2)$. Now assuming

$$A(r, t) = \frac{2}{1-r^2} \int_{S_r} |\omega_t|^2 ds \tag{3.5}$$

we get

$$\|\omega_t\|_p^2 = \int_0^p \frac{2}{1-r^2} dr \int_{S_r} |\omega_t|^2 ds = \int_0^p A(r, t) dr \tag{3.6}$$

and

$$\begin{aligned} \frac{d}{dt} \|\omega_t\|_p^2 &= \frac{d}{dt} t^{2p-2n} \int_{|z| \leq t\rho} [|\omega^1|^2 \frac{(1-r^2/t^2)^p}{(1-r^2)^p} \\ &+ |\omega^2|^2 \frac{(1-r^2/t^2)^{p+1}}{(1-r^2)^{p+1}} + |\omega^3|^2 \frac{(1-r^2/t^2)^{p+2}}{(1-r^2)^{p+2}}] G_{z/t} dz d\bar{z} \end{aligned} \tag{3.7}$$

where

$$G_{z/t} = i^{n^2} \det \left(\frac{4}{(1-r^2)^2} [(1-r^2/t^2) \delta_{\alpha\beta} + \bar{z}_\alpha z_\beta / t^2] \right) = i^{n^2} 4^n / (1-r^2/t^2)^{n+1} . \tag{3.8}$$

Then at $t=1$

$$\begin{aligned} \frac{d}{dt} \|\omega_t\|_p^2 &= \rho A(\rho, 1) + 2(p-n) \|\omega\|_p^2 - \int_{|z| \leq \rho} [(n+1-p) |\omega^1|^2 + \\ &(n-p) |\omega^2|^2 + (n-p-1) |\omega^3|^2] \frac{2r^2}{1-r^2} dV \\ &< \rho A(\rho, 1) - 2(n-p) \int_{|z| \leq \rho} |\omega|^2 \left(1 + \frac{(n-p-1)r^2}{(n-p)(1-r^2)} \right) dV \end{aligned} \tag{3.9}$$

Here we assume $p < n-1$ because the case of $p = n-1$ is simpler than the former and can be treated by the same way. It follows from Lemma 3.1 that

$$\rho A(\rho, 1) > 2(n - \rho) \int_{|z| \leq \rho} |\omega|^2 \left(1 + \frac{(n - p - 1)r^2}{(n - p)(1 - r^2)}\right) dV. \quad (3.10)$$

Suppose that $\omega \in A^p(F)$ do not equal zero identically on U^n . Thus there exists ρ_0 such that $0 < \rho_0 < 1$ and

$$K_0 \equiv 2(n - p) \int_{|z| \leq \rho_0} |\omega|^2 \left(1 + \frac{(n - p - 1)r^2}{(n - p)(1 - r^2)}\right) dV > 0. \quad (3.11)$$

Therefore

$$\rho A(\rho, 1) > K_0 + 2(n - p) \int_{\rho_0}^{\rho} \left(1 + \frac{(n - p - 1)r^2}{(n - p)(1 - r^2)}\right) A(r, 1) dr. \quad (3.12)$$

According to the method in [7], one can prove that

$$A(\rho, 1) > K_1 / (1 - \rho^2)^{n-p-1} \quad (3.13)$$

for some $K_1 > 0$ and all $\rho_0 < \rho < 1$. Then for any positive continuous function $f(r)$ with

$$\int_0^1 \frac{f(r)}{1 - r} dr = \infty$$

we have

$$\int_{U^n} f(r) e_{\omega} dV > \int_{\rho_0}^1 r f(r) A(r, 1) dr > K_1 \int_{\rho_0}^1 \frac{f(r)}{1 - r^2} dr = \infty.$$

This ends the proof.

Finally, instead of considering the complex hyperbolic space form functioning as the underlying manifolds, we can consider the other kind of underlying manifolds such as the Euclidean n -space R^n , the complex n -space C^n and some conformal flat Riemannian manifolds. Since the corresponding lemmas still exist, one may derive the corresponding estimations over these manifolds. For sake of simplicity, we only state the following.

THEOREM 3.1. Let ω be a p -form with values in F over C^n . If the stress energy tensor of ω is conservative and the norm of ω over C^n is finite or slowly divergent, then ω vanishes.

DEFINITION 3.1. A conformal flat Riemannian metric g on R^n is called to be of type A if

$$(A) \quad g^2 = f^2(x) \sum_{i=1}^n x_i^2 \text{ and the mean curvature normal } H \text{ of } S_r \text{ never}$$

pointing away from zero. g is called to be of type B if the underlying space is the open unit disc in R^n and

$$(B) \quad \tilde{g} = \tilde{f}^2(r) \sum_{i=1}^n dx_i^2 \text{ where } f \text{ satisfying } \int_0^1 \tilde{f}^{n-2p} dr = \infty \text{ and the mean}$$

curvature normal \tilde{H} of S_r never pointing away from zero.

THEOREM 3.2. Let M be a conformal flat Riemannian manifold of typed A or B. Let ω be a Riemannian vector bundle valued p -form over M , $p > n/2$, with conservative stress energy tensor. If the norm of ω over M is finite or slowly divergent. Then ω vanishes.

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