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CONVERGENCE OF THE SOLUTIONS FOR THE EQUATION $x^{(iv)} + a\dddot{x} + b\ddot{x} + g(\dot{x}) + h(x) = p(t,x,\dot{x},\ddot{x},\dddot{x})$

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ABSTRACT. This paper is concerned with differential equations of the form

$$x^{(iv)} + a\dddot{x} + b\ddot{x} + g(\dot{x}) + h(x) = p(t,x,\dot{x},\ddot{x},\dddot{x})$$

where a, b are positive constants and the functions g, h and p are continuous in their respective arguments, with the function h not necessarily differentiable. By introducing a Lyapunov function, as well as restricting the incrementary ratio $\eta^{-1}\{h(\xi + \eta) - h(\xi)\}$, $(\eta \neq 0)$, of h to a closed sub-interval of the Routh-Hurwitz interval, we prove the convergence of solutions for this equation. This generalizes earlier results.

KEY WORDS AND PHRASES. Routh-Hurwitz interval, Lyapunov function.
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1. INTRODUCTION.

Consider fourth-order differential equations of the form:

$$x^{(iv)} + a\dddot{x} + b\ddot{x} + g(\dot{x}) + h(x) = p(t,x,\dot{x},\ddot{x},\dddot{x}) \tag{1.1}$$

in which $a > 0$, $b > 0$, functions g and h are continuous in their respective arguments. The function $p(t,x,\dot{x},\ddot{x},\dddot{x})$ is assumed to have the form $q(t) + r(t,x,\dot{x},\ddot{x},\dddot{x})$ with the functions q and r depending explicitly on the arguments displayed, and continuous in their respective arguments. Further, we shall assume that $r(t,0,0,0,0) = 0$ for all t.

The solutions of (1.1) will be said to converge if any two solutions $x_1(t)$, $x_2(t)$ of (1.1) satisfy

$$\begin{aligned} x_2(t) - x_1(t) &\to 0, \ \dot{x}_2(t) - \dot{x}_1(t) \to 0 \\ \ddot{x}_2(t) - \ddot{x}_1(t) &\to 0, \ \dddot{x}_2(t) - \dddot{x}_1(t) \to 0, \end{aligned} \tag{1.2}$$

as $t \to \infty$.

The convergence of solutions for equations of the form (1.1) was earlier shown in [1], when $g(\dot{x}) = c\dot{x}$, with $c > 0$, and with the assumption that $h(x)$ is not necessarily differentiable, but with an incrementary ratio $\eta^{-1}\{h(\xi + \eta) - h(\xi)\}$,

$(\eta \neq 0)$, lying in a closed sub-interval I_o of the Routh-Hurwitz interval $(0, (ab - c)c/a^2)$, where

$$I_o \equiv \left[\Delta_o \, , \, \frac{K(ab - c)c}{a^2}\right] \tag{1.3}$$

$\Delta_o > 0$ and $K < 1$.

The main purpose of the present investigation is to give fourth-order analogues of [2], as well as extending earlier results in [1] to equations of the form (1.1) with the additional condition that for $y_1 \neq y_2$,

$$c_o \geqslant \frac{g(y_2) - g(y_1)}{y_2 - y_1} \geqslant c \tag{1.4}$$

for some constants $c_o > 0$ and $c > 0$, satisfying

$$abc > c_o^2 . \tag{1.5}$$

Moreover, while proving the convergence results for (1.1), we shall give a general estimate for the constant $K < 1$, from which a particular case is derived.

2. MAIN RESULTS.

The main results of this paper, which are in some respects fourth-order analogues of [2] and generalizations of [1], are the following:

THEOREM 1. Suppose that $g(0) = h(0)$ and that

(i) there are constants $c > 0$, $c_o > 0$ such that $g(y)$ satisfies inequalities (1.4) and (1.5);

(ii) there are constants $\Delta_o > 0$, $K < 1$ such that for any ξ, η, $(\eta \neq 0)$, the incrementary ratio for h satisfies

$$\eta^{-1}\{h(\xi + \eta) - h(\xi)\} \text{ lies in } I_o \tag{2.1}$$

with I_o as defined in (1.3);

(iii) there is a continuous function $\phi(t)$ such that

$$\begin{aligned} &|r(t,x_2,y_2,z_2,w_2) - r(t,x_1,y_1,z_1,w_1)| \\ &\leqslant \phi(t)\{|x_2-x_1| + |y_2-y_1| + |z_2-z_1| + |w_2-w_1|\} \end{aligned} \tag{2.2}$$

holds for arbitrary t, x_1, y_1, z_1, w_1, x_2, y_2, z_2, and w_2.

Then, there exists a constant D_1 such that if

$$\int_o^t \phi^\alpha(\tau)d\tau \leqslant D_1 t \tag{2.3}$$

for some α, in the range $1 \leqslant \alpha \leqslant 2$, then all solutions of (1.1) converge.

A very important step in the proof of Theorem 1 will be to give estimates for any two solutions of (1.1). This in itself, being of independent interest, is given as:

THEOREM 2. Let $x_1(t)$, $x_2(t)$ be any two solutions of (1.1). Suppose that all the conditions of Theorem 1 are satisfied, then for each fixed α, in the range $1 \le \alpha \le 2$, there exist constants D_2, D_3 and D_4 such that for $t_2 \ge t_1$,

$$S(t_2) \le D_2 S(t_1) \exp\{-D_3(t_2-t_1) + D_4\int_{t_1}^{t_2} \phi^{\alpha}(\tau)d\tau\} \tag{2.4}$$

where

$$S(t) = \{[x_2(t)-x_1(t)]^2 + [\dot{x}_2(t)-\dot{x}_1(t)]^2 + [\ddot{x}_2(t)-\ddot{x}_1(t)]^2 + [\dddot{x}_2(t)-\dddot{x}_1(t)]^2\}. \tag{2.5}$$

If we put $x_1(t) = 0$ and $t_1 = 0$, we immediately obtain:

COROLLARY 1. If $p = 0$ and the hypotheses (i) and (ii) of Theorem 1 hold, then the trivial solution of (1.1) is exponentially stable in the large.

Further, if we put $\xi = 0$ in (2.1) with η ($\eta \ne 0$) arbitrary, we obtain:

COROLLARY 2. If $p = 0$ and the hypotheses (i) and (ii) hold for arbitrary η ($\eta \ne 0$), and $\xi = 0$, then there exists a constant $D_5 > 0$ such that every solution $x(t)$ of (1.1) satisfies

$$|x(t)| \le D_5; \; |\dot{x}(t)| \le D_5; \; |\ddot{x}(t)| \le D_5; \; |\dddot{x}(t)| \le D_5. \tag{2.6}$$

3. PRELIMINARY RESULTS.

Let $Q(t) = \int_0^t q(\tau)d\tau$. For convenience, by setting $\dot{x} = y$, $\dot{y} = z$ and $\dot{z} = w + Q(t)$, we replace equation (1.1) by the equivalent system:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= w + Q(t) \\ \dot{w} &= -aw - bz - g(y) - h(x) + r(t,x,y,z,w+Q(t)) - aQ(t) \end{aligned} \tag{3.1}$$

Let $(x_i(t), y_i(t), z_i(t), w_i(t))$, $(i = 1,2)$, be two solutions of (3.1), such that

$$c \le \frac{g(y_2) - g(y_1)}{y_2 - y_1} \le c_o \; ; \tag{3.2}$$

and

$$\Delta_o \le \frac{h(x_2) - h(x_1)}{x_2 - x_1} \le \frac{K(ab - c)c}{a^2} \tag{3.3}$$

where c, c_0, Δ_0, K are as defined in (1.3), (1.4) and (1.5).

Our main tool in the proofs of the convergence Theorems will be the following function: $W = W(x_2 - x_1, y_2 - y_1, x_2 - z_1, w_2 - w_1)$ defined by

$$\begin{aligned} 2W = \{&c^2\varepsilon(1-\varepsilon)(x_2-x_1)^2 + ac(1-\varepsilon)(D-1)(y_2-y_1)^2 + \\ &+ 2c[\varepsilon + (D-1)](y_2-y_1)(z_2-z_1) + \varepsilon D(w_2-w_1)^2 + \\ &+ b(D-1)(z_2-z_1)^2 + [(1-\varepsilon)D - 1][a(z_2-z_1)+(w_2-w_1)]^2 + \\ &+ [c(1-\varepsilon)(x_2-x_1) + b(y_2-y_1) + (z_2-z_1) + (w_2-w_1)]^2\}, \end{aligned} \tag{3.4}$$

where $D-1 = (\delta + c\varepsilon)/(ab - c - \delta)$, with $ab - c > \delta > 0$; $0 < \varepsilon < 1$; and $ab\varepsilon(2-\varepsilon) = \delta$. This is an adaptation of the function V used in [1].

Since $0 < \varepsilon < 1$, following the argument used in [1], we can easily verify the following for W.

LEMMA 1. (i) $W(0,0,0,0) = 0$; and

(ii) there exist finite constants $D_6 > 0$, $D_7 > 0$ such that

$$D_6\{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 + (w_2-w_1)^2\} \leq W \leq \leq D_7\{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 + (w_2-w_1)^2\}. \tag{3.5}$$

If we define the function $W(t)$ by $W(x_2(t) - x_1(t),\ y_2(t) - y_1(t),\ z_2(t) - z_1(t),\ w_2(t) - w_1(t))$, and using the fact that the solutions $(x_i, y_i, z_i, w_i + Q(t))$, $(i = 1,2)$, satisfy (3.1), then $S(t)$ as defined in (2.5) becomes

$$S(t) = \{[x_2(t)-x_1(t)]^2 + [y_2(t)-y_1(t)]^2 + [z_2(t)-z_1(t)]^2 + [w_2(t)-w_1(t)]^2\} \tag{3.6}$$

We can then prove the following result on the derivative of $W(t)$ with respect to t.

LEMMA 2. Let the hypotheses (i) and (ii) of Theorem 1 hold. Then, there exist positive finite constants D_8 and D_9 such that

$$\frac{dW}{dt} \leq -2D_8S + D_9S^{\frac{1}{2}}|\Theta| \tag{3.7}$$

where $\Theta = r(t,x_2,y_2,z_2,w_2 + Q) - r(t,x_1,y_1,z_1,w_1 + Q)$.

PROOF OF LEMMA 2. On using (3.1), a direct computation of $\frac{dW}{dt}$ gives after simplification

$$\frac{dW}{dt} = -W_1 + W_2 \tag{3.8}$$

where

$$\begin{aligned} W_1 = {} & \{c(1-\varepsilon)H(x_2,x_1)(x_2-x_1)^2 + bc\varepsilon(y_2-y_1)^2 + ab\varepsilon(1-\varepsilon)D(z_2-z_1)^2 \\ & + a\varepsilon D(w_2-w_1)^2\} + \{G(y_2,y_1) - c\}\{c(1-\varepsilon)(x_2-x_1) + \\ & + b(y_2-y_1) + a(1-\varepsilon)D(z_2-z_1) + D(w_2-w_1)\}(y_2-y_1) + \\ & + H(x_2,x_1)\{b(y_2-y_1) + a(1-\varepsilon)D(z_2-z_1) + D(w_2-w_1)\}(x_2-x_1) \end{aligned}$$

and

$$W_2 = \Theta(t)\{c(1-\varepsilon)(x_2-x_1) + b(y_2-y_1) + a(1-\varepsilon)D(z_2-z_1) + D(w_2-w_1)\},$$

with

$$G(y_2,y_1) = \frac{g(y_2) - g(y_1)}{y_2 - y_1}, \quad (y_2 \neq y_1); \tag{3.9}$$

$$H(x_2,x_1) = \frac{h(x_2) - h(x_1)}{x_2 - x_1}, \quad (x_2 \neq x_1). \tag{3.10}$$

Let $\lambda = \{G(y_2,y_1) - c\} > 0$ for $y_2 \neq y_1$. Define

$$\sum_{i=1}^{5}\alpha_i = 1 \;;\; \sum_{i=1}^{5}\beta_i = 1 \;;\; \sum_{j=1}^{3}\gamma_j = 1 \text{ and } \sum_{j=1}^{3}\delta_j = 1,$$

with $\alpha_i > 0$, $\beta_i > 0$, $\gamma_j > 0$ and $\delta_j > 0$. Further, let us denote $H(x_2,x_1)$ simply by H. Then, we can re-arrange W_1 as

$$W_1 = W_{11} + W_{12} + W_{13} + W_{14} + W_{21} + W_{23} + W_{24} \tag{3.11}$$

where

$$W_{11} = \{\alpha_1 c(1-\varepsilon)H(x_2-x_1)^2 + b(\beta_1 c\varepsilon + \lambda)(y_2-y_1)^2 + \gamma_1 ab\varepsilon(1-\varepsilon)D(z_2-z_1)^2 + \delta_1 a\varepsilon D(w_2-w_1)^2\}$$

$$W_{12} = \{\beta_2 bc\varepsilon(y_2-y_1)^2 + \lambda c(1-\varepsilon)(x_2-x_1)(y_2-y_1) + + \alpha_2 c(1-\varepsilon)H(x_2-x_1)^2\} \;;$$

$$W_{23} = \{\beta_3 bc\varepsilon(y_2-y_1)^2 + \lambda a(1-\varepsilon)D(y_2-y_1)(z_2-z_1) + + \gamma_2 ab\varepsilon(1-\varepsilon)D(z_2-z_1)^2\} \;;$$

$$W_{24} = \{\beta_4 bc\varepsilon(y_2-y_1)^2 + \lambda D(y_2-y_1)(w_2-w_1) + \delta_2 a\varepsilon D(w_2-w_1)^2\};$$

$$W_{12} = \{\alpha_3 c(1-\varepsilon)H(x_2-x_1)^2 + bH(x_2-x_1)(y_2-y_1) + \beta_5 bc\varepsilon(y_2-y_1)^2\} \;;$$

$$W_{13} = \{\alpha_4 c(1-\varepsilon)H(x_2-x_1)^2 + a(1-\ )DH(x_2-x_1)(z_2-z_1) + \beta_3 ab\varepsilon(1-\varepsilon)D(z_2-z_1)^2\} \;;$$

and

$$W_{14} = \{\alpha_5 c(1-\varepsilon)H(x_2-x_1)^2 + DH(x_2-x_1)(w_2-w_1) +$$

Each W_{ij}, $(i \neq j)$, $(i = 1,2;\ j = 1,2,3,4)$, is quadratic in its respective variables. Also, using the fact that any quadratic of the form $Au^2 + Buv + Cv^2$ is non-negative if $(4AC - B^2) > 0$, we obtain that

$$W_{21} \geq 0 \quad \text{if} \quad \lambda^2 \leq \frac{4b\varepsilon\Delta_o\alpha_2\beta_2}{1-\varepsilon} \;;$$

$$W_{23} \geq 0 \quad \text{if} \quad \lambda^2 \leq \frac{4b^2c\varepsilon^2\beta_3\gamma_2}{a(1-\varepsilon)D} \;;$$

$$W_{24} \geq 0 \quad \text{if} \quad \lambda^2 \leq \frac{4abc\varepsilon^2\delta_2\beta_4}{D} \;;$$

$$W_{12} \geq 0 \quad \text{if} \quad H \leq \frac{4c^2\varepsilon(1-\varepsilon)\alpha_3\beta_5}{b} \;;$$

$$W_{13} \geq 0 \quad \text{if} \quad H \leq \frac{4bc\varepsilon\alpha_4\gamma_3}{aD} \;;$$

and

$$W_{14} \geq 0 \quad \text{if} \quad H \leq \frac{4ac\varepsilon(1-\varepsilon)\alpha_5\delta_3}{D} .$$

Thus $W_1 \geq W_{11}$, provided that

$$0 \leq \lambda^2 \leq 4\min\left\{\frac{b\varepsilon\Delta_o\alpha_2\beta_2}{(1-\varepsilon)};\ \frac{b^2c\varepsilon\beta_3\gamma_2}{a(1-\varepsilon)D};\ \frac{abc\varepsilon^2\delta_2\beta_4}{D}\right\} \tag{3.12}$$

and

$$H \text{ lies in } I_o \equiv \left[\Delta_o,\ \frac{K(ab-c)c}{a^2}\right] \tag{3.13}$$

a closed sub-interval of the Routh-Hurwitz interval $(0,(ab-c)c/a^2)$, with

$$K = \left(\frac{4}{ab-c}\right)\min\left\{\frac{ca^2\varepsilon(1-\varepsilon)\alpha_3\beta_5}{b};\ \frac{ab\varepsilon\alpha_4\gamma_3}{D};\ \frac{a^3\varepsilon(1-\varepsilon)\alpha_5\delta_3}{D}\right\} \tag{3.14}$$
$$< 1 .$$

By choosing $2D_8 = \min\{c(1-\varepsilon)\Delta_0;\ bc\varepsilon;\ ab\varepsilon\ (1-\varepsilon)D;\ a\varepsilon D\}$, we clearly have

$$W_1 \geq W_{11} \geq 2D_8S . \tag{3.15}$$

also, if we choose $D_9 = 2\max\{c(1-\varepsilon);\ b;\ a(1-\varepsilon)D;\ D\}$, we obtain:

$$W_2 \leq D_9S^{\frac{1}{2}}|\Theta| . \tag{3.16}$$

Combining (3.15) and (3.16) in (3.8), we obtain (3.7). This completes the proof of Lemma 2.

4. PROOF OF THEOREM 2.

This follows directly from [3], on using inequality (3.7). Let α be any constant in the range $1 < \alpha < 2$. Set $2\mu = 2-\alpha$, so that $0 < 2\mu < 1$. We re-write (3.7) in the form

$$\frac{dW}{dt} + D_8S \leq D_9S^{\mu}W^*$$

where
$$W^* = (|\Theta| - D_8D_9^{-1}S^{\frac{1}{2}})S^{\frac{1}{2}-\mu} . \tag{4.1}$$

Considering the two cases (i) $|\Theta| < D_8S^{1/2}/D_9$ and (ii) $|\Theta| > D_8S^{1/2}/D_9$ separately, we find that in either case, there exists some constant $D_{11} > 0$ such that $W^* < D_{11}|\Theta|^{2(1-\mu)}$. Thus using (2.2), inequality (4.1) becomes

$$\frac{dW}{dt} + D_8S \leq D_{12}S^{\mu}\phi^{2(1-\mu)}S^{(1-\mu)} , \tag{4.2}$$

where $D_{12} \geq 2D_9D_{11}$. This immediately gives

$$\frac{dW}{dt} + (D_{13} - D_{14}\phi^{\alpha}(t))W \leq 0 \tag{4.3}$$

after using Lemma 1 on W, with D_{13} and D_{14} as some positive constants.

On integrating (4.3) from t_1 to t_2, $(t_2 \geq t_1)$, we obtain

$$W(t_2) \leq W(t_1) \exp \{-D_{13}(t_2-t_1) + D_{14}\int_{t_1}^{t_2} \phi^{\alpha}(\tau)d\tau\}. \tag{4.4}$$

Again, using Lemma 1, we obtain (2.4), with $D_2 = D_7/D_6$, $D_3 = D_{13}$ and $D_4 = D_{14}$. This completes the proof of Theorem 2.

5. PROOF OF THEOREM 1.

This follows from the estimate (2.4) and the condition (2.3) on $\phi(t)$. Choose $D_1 = D_3/D_4$ in (2.3). Then, as $t = (t_2-t_1) \to \infty$, $S(t) \to 0$, which proves that as $t \to \infty$,

$$x_2(t)-x_1(t) \to 0, \; \dot{x}_2(t)-\dot{x}_1(t) \to 0,$$
$$\ddot{x}_2(t)-\ddot{x}_1(t) \to 0, \; \dddot{x}_2(t)-\dddot{x}_1(t) \to 0.$$

This completes the proof of Theorem 1.

6. REMARKS.

(i) If in (3.14) we choose

$$\alpha_1 = 1/2 \;;\; \alpha_j = 1/8 \quad (j = 2,3,4,5);$$
$$\beta_1 = 1/2 \;;\; \beta_j = 1/8 \quad (j = 2,3,4,5);$$
$$\gamma_1 = 1/2 \;;\; \gamma_2 = \gamma_3 = 1/4 \;;$$
$$\delta_1 = 1/2 \;;\; \delta_2 = \delta_3 = 1/4 \;,$$

we obtain

$$K = \left(\frac{1}{16(ab-c)}\right) \min \left\{ \frac{ca^2\varepsilon(1-\varepsilon)}{b} \;;\; \frac{2ab\varepsilon}{D} \;;\; \frac{2a^3\varepsilon(1-\varepsilon)}{D} \right\} < 1.$$

(ii) As remarked in [1], the results remain valid if we replace $\phi(t)$ in (2.3) by a constant $D_{15} > 0$.

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