Internat. J. Math. & Math. Sci.
VOL. 12 NO. 1 (1989) 99-106

ON (J,p_n) SUMMABILITY OF FOURIER SERIES

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(Received December 13, 1986 and in revised form April 8, 1987)

ABSTRACT. In this note two theorems have been established. The first one deals with the summability (J,p_n) of a Fourier series while the second on concerns with the summability of the first derived Fourier series. These results include, as a special case, certain results of Nanda [1].

KEY WORDS AND PHRASES. (J,p_n) summability, Fourier series, Derived Fourier series.
1980 AMS SUBJECT CLASSIFICATION CODES. 42A24, 40G10.

INTRODUCTION.

Let $\{p_n\}$ be a sequence of non-negative numbers such that $\sum_o^\infty p_n$ diverges and let the radius of convergence of power series

$$p(z) = \sum_o^\infty p_n x^n \tag{1.1}$$

be 1. Given any series $\Sigma\, a_n$ with the sequence of partial sums $\{s_n\}$ we write

$$p_s(x) = \sum_o^\infty p_n s_n x^n \tag{1.2}$$

and

$$J_s(x) = \frac{p_s(x)}{p(x)} \; . \tag{1.3}$$

If the series in (1.2) is convergent in [0,1) and

$$\lim_{x\to 1-} J_s(x) = s,$$

we say that the series $\Sigma\, a_n$ or the sequence $\{s_n\}$ is summable (J,p_n) to s, where s is a finite number. ([2], [3], p.80).
For $p_n=1$, $\frac{1}{n}$ with $p_o=0$, and A_n^k, $k > -1$ we get summability A, summability (L) and A_k method of summability respectively.

Suppose f is a periodic function with period 2π and integrable in the sense of Lebesgue over $(-\pi,\pi)$. Let

$$f(x) \sim \frac{a_0}{2} + \sum_{1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{o}^{\infty} A_n(x). \tag{1.4}$$

Then the first derived series of (1.4) is

$$\sum_{1}^{\infty} n(b_n \cos nx - a_n \sin nx) = \sum_{1}^{\infty} n\, B_n(x). \tag{1.5}$$

We write

$$\phi(t) = \frac{1}{2} \{f(x_0 + t) + f(x_0 - t) - 2s\}$$

$$\psi(t) = \frac{1}{2} \{f(X_0 + t) - f(X_0 - t)\}$$

$$g(t) = \frac{\psi(t)}{2\sin t/2} - s$$

$$M(t) = \sum_{o}^{\infty} p_n x^n \sin nt$$

$$\Phi(t) = \int_t^{\pi} \frac{\phi(u)}{u}\, du$$

and

$$G(t) = \int_t^{\pi} \frac{g(u)}{u}\, du.$$

2. MAIN RESULTS.

In this note we propose to establish the following theorems on (J,p_n) summability of (1.4) and (1.5).

THEOREM 1. Let $\{p_n\}$ be a positive sequence such that

(a) $$n\, p_n = O(1), \sum_{k=\nu}^{\infty} \frac{p_k}{k+1} = O(p_\nu) \text{ and}$$

(b) $$\sum_{n=0}^{\infty} |\Delta^2(n\, p_n\, x^n)| = O(1-x), \quad 0 < x < 1.$$

If $\Phi(t) = o(p(1-t))$, $t \to +0$. then the Fourier series (1.4) is summable (J,p_n) to s.

THEOREM 2. Let $\{p_n\}$ satisfy the hypothesis (a) of Theorem 1. If

(c) $$\sum_{o}^{\infty} n|\Delta^3 (n\, p_n\, x^n)| = O(1-x), \; 0 < x < 1$$

and $G(t) = o(p(1-t))$, as $t \to 0+$, then the first derived series (1.5) is summable (J,p_n) to s.

It may be remarked that for $p_n = \frac{1}{n}$, $n \geq 1$, $p_o = 0$ our theorems include two known theorems of Nanda [1] on L-summability of Fourier series and its derived series. For an earlier result on (J,p_n) summability of (1.4) under more stringent conditions see

Khan [4]. Very recently in 1985 Prem Chandra, Mohapatra and Sahney [5] have established a similar theorem on (J,p_n) summability with another set of conditions. It may be observed that $n\,p_n = 0(1)$ $p'(x) = 0(\frac{1}{1-x})$ as $x \to 1-$. Also it is easy to see that $n\,p_n < (n+1)\,p_{n+1}$, $n = 0,1,2,\ldots$ and $p'(x) = 0(\frac{1}{1-x})$ imply that $n\,p_n = 0(1)$. For if $\{n\,p_n\}$ is not bounded then $\lim_{n\to\infty} n\,p_n = \infty$. Now using the well known result that

$$\lim_{n\to\infty} \frac{\beta_n}{\alpha_n} = \infty \quad \lim_{x\to 1-} \frac{\sum_o^\infty \beta_n x^n}{\sum_o^\infty \alpha_n x^n} = \infty,$$

where radius of convergence of each power series is 1 and $\alpha > 0$ with

$\sum_o^\infty \alpha_n = \infty$, we find for $\alpha_n = 1$, $\beta_n = n\,p_n$ that

$\lim_{x\to 1-} (1-x) \sum_o^\infty n\,p_n\,x^n = \infty$. This means $\lim_{x\to 1-} (1-x)p'(x) = \infty$ which contradicts the hypothesis that $(1-x)p'(x) = 0(1)$ as $x \to 1-$. Thus conditions (2.1) and (2.4) of [5] imply that $n\,p_n = 0(1)$.

3. PROOF OF THEOREM 1.

Let $s_n(x_0)$ denote the n-th partial sum of (1.4) at $x = x_0$. Then

$$s_n(x_0) - s = \frac{2}{\pi} \int_0^\pi \phi(t)\,\frac{\sin nt}{t}\,dt + o(1)$$

so that

$$\sum_{n=0}^\infty p_n\,x^n(s_n(x_0) - s) = \frac{2}{\pi} \int_0^\pi \frac{\phi(t)}{t} \sum_{n=0}^\infty p_n\,x^n \sin nt\,dt + o(p(x))$$

$$= \frac{2}{\pi}\int_0^\pi - \Phi'(t)M(t)\,dt + o(p(x))$$

$$= \frac{2}{\pi}\int_0^\pi \Phi(t) \sum_o^\infty n\,p_n x^n \cos nt\,dt + o(p(x))$$

$$= \frac{2}{\pi}\left(\int_0^{1-x} + \int_{1-x}^\pi\right)\ldots + o(p(x))$$

$$= I_1 + I_2 + o(p(x)), \text{ say.} \tag{3.1}$$

Now

$$I_1 = \frac{2}{\pi}\int_0^{1-x} \Phi(t) \sum_{n=0}^\infty n\,p_n x^n \cos nt\,dt$$

$$= o\left(\int_0^{1-x} o(p(1-t))\,\frac{1}{1-x}\,dt\right) \tag{3.2}$$

$$= o\left(\frac{1}{1-x}\right)\int_0^{1-x} \sum_{n=0}^\infty p_k\,(1-t)^k\,dt$$

Khan [4]. Very recently in 1985 Prem Chandra, Mohapatra and Sahney [5] have established a similar theorem on (J,p_n) summability with another set of conditions. It may be observed that $n\,p_n = O(1)$ $p'(x) = O(\frac{1}{1-x})$ as $x \to 1-$. Also it is easy to see that $n\,p_n < (n+1)\,p_{n+1}$, $n = 0,1,2,\ldots$ and $p'(x) = O(\frac{1}{1-x})$ imply that $n\,p_n = O(1)$. For if $\{n\,p_n\}$ is not bounded then $\lim_{n\to\infty} n\,p_n = \infty$. Now using the well known result that

$$\ln_{n\to\infty} \frac{\beta_n}{\alpha_n} = \infty \quad \ln_{x\to 1-} \frac{\sum_{o}^{\infty} \beta_n x^n}{\sum_{o}^{\infty} \alpha_n x^n} = \infty,$$

where radius of convergence of each power series is 1 and $\alpha > 0$ with

$$\sum_{o}^{\infty} \alpha_n = \infty, \text{ we find for } \alpha_n = 1,\ \beta_n = n\,p_n \text{ that}$$

$\lim_{x\to 1-} (1-x) \sum_{o}^{\infty} n\,p_n\,x^n = \infty$. This means $\lim_{x\to 1-} (1-x)p'(x) = \infty$ which contradicts the hypothesis that $(1-x)p'(x) = O(1)$ as $x \to 1-$. Thus conditions (2.1) and (2.4) of [5] imply that $n\,p_n = O(1)$.

3. PROOF OF THEOREM 1.

Let $s_n(x_0)$ denote the n-th partial sum of (1.4) at $x = x_0$. Then

$$s_n(x_0) - s = \frac{2}{\pi} \int_0^{\pi} \phi(t)\, \frac{\sin nt}{t}\, dt + o(1)$$

so that

$$\sum_{n=0}^{\infty} p_n\, x^n (s_n(x_0) - s) = \frac{2}{\pi} \int_0^{\pi} \frac{\phi(t)}{t} \sum_{n=0}^{\infty} p_n\, x^n \sin nt\, dt + o(p(x))$$

$$= \frac{2}{\pi} \int_0^{\pi} - \Phi'(t)M(t)\, dt + o(p(x))$$

$$= \frac{2}{\pi} \int_0^{\pi} \Phi(t) \sum_{o}^{\infty} n\, p_n x^n \cos nt\, dt + o(p(x))$$

$$= \frac{2}{\pi} \left(\int_0^{1-x} + \int_{1-x}^{\pi} \right) \ldots + o(p(x))$$

$$= I_1 + I_2 + o(p(x)), \text{ say.} \tag{3.1}$$

Now

$$I_1 = \frac{2}{\pi} \int_0^{1-x} \Phi(t) \sum_{n=0}^{\infty} n\, p_n x^n \cos nt\, dt$$

$$= o\left(\int_0^{1-x} o(p(1-t))\, \frac{1}{1-x}\, dt \right) \tag{3.2}$$

$$= o\left(\frac{1}{1-x}\right) \int_0^{1-x} \sum_{n=0}^{\infty} p_k\, (1-t)^k\, dt$$

$$= \frac{o(1)}{1-x} \sum_{k=0}^{\infty} \frac{p_k}{k+1} \left(1-x^{k+1}\right) = o(1) \sum_{k=0}^{\infty} \frac{p_k}{k+1} \sum_{\nu=0}^{k} x^{\nu}$$

$$= o(1) \sum_{\nu=0}^{\infty} x^{\nu} \sum_{k=\nu}^{\infty} \frac{p_k}{k+1} = o(1) \sum_{\nu=0}^{\infty} p_{\nu} x^{\nu} = o(p(x)).$$

Thus

$$I_1 = o(p(x)). \tag{3.3}$$

Again

$$I_2 = \frac{2}{\pi} \int_{1-x}^{\pi} \Phi(t) \sum_{k=0}^{\infty} k\, p_k\, x^k \cos kt$$

$$= \frac{2}{\pi} \int_{1-x}^{\pi} \Phi(t) \sum_{k=0}^{\infty} \Delta^2 (k\, p_k\, x^k)\, F_k(t)$$

where

$$F_k(t) = \sum_{\nu=0}^{k} D_{\nu}(t) = \sum_{\nu=0}^{k} D_{\nu}(t) = \sum_{\nu=0}^{k} \frac{\sin(\nu+\frac{1}{2})t}{2 \sin t/2} .$$

Under the hypothesis of Theorem 1

$$I_2 = O(1) \int_{1-x}^{\pi} p(1-t) \frac{(1-x)}{t^2}\, dt$$

$$= o(p(x)) \int_{1-x}^{\pi} \frac{(1-x)\, dt}{t^2}$$

$$= o(p(x)) . \tag{3.4}$$

Thus in view of (3.1), (3.3) and (3.4)

$$\sum_{o}^{\infty} p_n\, x^n\, (s_n(x_0) - s) = o(p(x)) \text{ as } x \to 1- .$$

This proves Theorem 1.

4. PROOF OF THEOREM 2. As shown in ([6], p. 54) we can assume that $s = 0$.

Let $T_n(x_0)$ denote the n-th partial sum of (1.5) at $x = x_0$. Then

$$T_n(x_0) = \frac{1}{\pi} \int_0^{\pi} \frac{g(t) \sin nt}{\sin t/2}\, dt - \frac{2n}{\pi} \int_0^{\pi} \cos\left(n + \tfrac{1}{2}\right) t\, g(t)\, dt$$

$$= \frac{2}{\pi} \int_0^{\pi} \frac{g(t) \sin nt}{t}\, dt + o(1) - \frac{2n}{\pi} \int_0^{\pi} \cos\left(n + \tfrac{1}{2}\right) t\, g(t)\, dt ,$$

$$= T_{n,1} + o(1) + T_{n,2} \text{ , say,}$$

so that

$$\sum_{o}^{\infty} T_n(x_0)\, p_n\, x^n = \sum_{o}^{\infty} T_{n1}\, p_n\, x^n + o(p(x)) + \sum_{o}^{\infty} T_{n2}\, p_n\, x^n$$

$$= L_1 + o(p(x)) + L_2 \text{ , say.} \tag{4.1}$$

As shown in the proof of Theorem 1 in view of (4.5), $L_1 = o(p(x))$. (4.2)

Let $H(t) = \sum_0^\infty n\, p_n\, x^n \cos(n + \frac{1}{2})t$. Then

$$L_2 = -\frac{2}{\pi} \sum_0^\infty n\, p_n\, x^n \int_0^\pi \cos(n+\tfrac{1}{2})t\, g(t)\, dt$$

$$= -\frac{2}{\pi} \int_0^\pi g(t) \sum_{n=0}^\infty n\, p_n\, x^n \cos(n+\tfrac{1}{2})\, t\, dt$$

$$= -\frac{2}{\pi} \int_0^\pi g(t)\, H(t)\, dt = -\frac{2}{\pi} \int_0^\pi -\, t\, G'(t) H(t)\, dt$$

$$= \frac{2}{\pi} \{[tG(t)\, H(t)]_0^\pi - \int_0^\pi G(t) \frac{d}{dt} (tH(t))\} dt$$

$$= -\frac{2}{\pi} \int_0^\pi G(t)\, \{\, H(t) + tH'(t)\}\, dt$$

$$= -\frac{2}{\pi} \left(\int_0^{1-x} + \int_{1-x}^\pi \right) = L_{21} + L_{22}, \text{ say.}$$

Now since $n\, p_n = O(1)$

$$L_{21} = \int_0^{1-x} \frac{o(p(1-t)}{1-x}\, dt$$

$$+ \int_0^{1-x} o(p(1-t))\, t \sum_0^\infty n\, x^n\, dt$$

$$= o(p(x)) + o(1) \int_0^{1-x} \frac{p(1-t)}{1-x}\, dt = o(p(x)) \tag{4.3}$$

as shown in (3.2).

In view of the hypothesis of Theorem 2.

$$L_{22} = -\frac{2}{\pi} \int_{1-x}^\pi o(p(1-t)) \left|\frac{d}{dt} (t\, H(t))\right| dt\ . \tag{4.4}$$

Since $n\, p_n = O(1)$, we have by using Abel's transformation

$$\frac{d}{dt}\, t\, H(t) = \sum_0^\infty n\, p_n\, x^n \frac{d}{dt}\, t \cos(n + \tfrac{1}{2})\, t$$

$$= \sum_0^\infty \Delta(n\, p_n\, x^n) \frac{d}{dt}\ \frac{t}{2 \sin t/2} (\sin\, (n+1)t)$$

$$= \sum_0^\infty \Delta^2(n\, p_n\, x^n) \frac{d}{dt} \left\{\frac{t}{4 \sin^2 t/2} \left(\cos t/2 - \cos(n + 3/2)t\right)\right\}.$$

Thus

$$\left|\frac{d}{dt}(t\, H(t)\right| < \frac{C}{t^2} \sum_{n=o}^\infty \left|\, \Delta^2\, n\, p_n\, x^n\right|$$

$$+ \frac{C}{t} \left| \sum_0^\infty n\, \Delta^2(n\, p_n\, x^n) \sin\, (n+3/2)t \right|$$

$$< \frac{C(1-x)}{t^2} + \frac{C}{t} \left| \sum_0^\infty \Delta\{n\Delta^2\, (n\, p_n\, x^n)\} \sum_{v=o}^n \sin(v+3/2)t \right|$$

$$< \frac{C(1-x)}{t^2} + \frac{C}{t^2} \Sigma\, n\left|\Delta^3(n\, p_n\, x^n)\right|$$

$$< \frac{C(1-x)}{t^2},$$

where C is a positive constant not necessarily the same at each occurrence, and in view of the fact that $\sum\limits_0^\infty n|\Delta^3(n\, p_n\, x^n) = O(1-x)$ (4.5)
implies that $\sum\limits_0^\infty |\Delta^2(n\, p_n\, x^n)| = O(1-x)$.
Hence from (4.4)

$$L_{22} = o(1) \int_{1-x}^{\pi} p(1-t) \frac{(1-x)}{t^2}\, dt = o(p(x)), \text{ as shown in (3.3).} \qquad (4.6)$$

Thus from (4.1) - (4.3) and (4.6) the proof of Theorem 2 follows.

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