Internat. J. Math. & Math. Sci.
VOL. 12 NO. 1 (1989) 137-144

APPLICATION OF DECOMPOSITION TO HYPERBOLIC, PARABOLIC, AND ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

G. ADOMIAN

Center for Applied Mathematics
University of Georgia
Athens, Georgia 30602

(Received September 22, 1987)

ABSTRACT: The decomposition method is applied to examples of hyperbolic, parabolic, and elliptic partial differential equations without use of linearization techniques. We consider first a nonlinear dissipative wave equation; second, a nonlinear equation modeling convection-diffusion processes; and finally, an elliptic partial differential equation.

KEY WORDS AND PHRASES. Decomposition method, linear and nonlinear partial differential equations, parabolic equations, elliptic equations, hyperbolic equations.
1980 AMS SUBJECT CLASSIFICATION CODES. 35L35, 35L70.

1. INTRODUCTION:

The decomposition method [1] has developed rapidly and is now providing solutions in the form of converging analytic series for broad categories of ordinary or partial differential equations or systems of equations with given initial/boundary conditions. An important advantage is that linearization is not required. The rather global character is shown by application to some typical examples - a dissipative wave equation, and equation modeling convection-diffusion processes, and an elliptic equation without the use of linearization techniques.

2. HYPERBOLIC CASE:

Consider the dissipative wave equation $u_{tt} - u_{xx} + (\partial/\partial t)f(u) = 0$ with $f(u(x,t))$ a continuous bounded function and $(t,x) \in [0,T] \times R$. Let $L_t = \partial^2/\partial t^2$ and $L_x = \partial^2/\partial x^2$ and write

$$L_t u - L_x u = - (\partial/\partial t)f(u(x,t)) \qquad (2.1)$$

Using the decomposition method [1] we solve for each linear term thus

$$L_t u = L_x u - (\partial/\partial t)f(u) \tag{2.2}$$

$$L_x u = L_t u + (\partial/\partial t)f(u)$$

Operating with the inverses, we have $L_t^{-1}L_t u = u - \phi_t$ and $L_x^{-1}L_x u = u - \phi_x$, where the homogeneous solutions are evaluated from the given initial/boundary conditions. Thus (2.2) becomes

$$u = \phi_t + L_t^{-1}L_x u - L_t^{-1}(\partial/\partial t)f(u)$$

$$u = \phi_x + L_x^{-1}L_t u + L_x^{-1}(\partial/\partial t)f(u)$$

Adding and dividing by two

$$u = (1/2)(\phi_t + \phi_x) + (1/2)(L_t^{-1}L_x + L_x^{-1}L_t)u$$

$$- (1/2)(L_t^{-1} - L_x^{-1})(\partial/\partial t)f(u)$$

or if

$$K = (1/2)(L_t^{-1}L_x + L_x^{-1}L_t)$$

$$G = - (1/2)(L_t^{-1} - L_x^{-1}) \tag{2.3}$$

$$u_0 = (1/2)(\phi_t + \phi_x)$$

we have

$$u = u_0 + Ku + G(\partial/\partial t)f(u) \tag{2.4}$$

a result also obtained by operating on (2.1) with $(L_t^{-1} - L_x^{-1})$.

Let $u = \sum_{n=0}^{\infty} u_n$ with u_0 as defined and let $f(u) = \sum_{n=0}^{\infty} A_n$, where the A_n are generated for the specific function f(u) as discussed in [1]. Now

$$u = u_0 + K \sum_{n=0}^{\infty} u_n + G(\partial/\partial t) \sum_{n=0}^{\infty} A_n \tag{2.5}$$

We define

$$u_{n+1} = Ku_n + G(\partial/\partial t)A_n \tag{2.6}$$

for $n > 0$ to complete the solution. An n-term approximation $\phi_n = \sum_{i=0}^{n-1} u_i$ suffices as discussed in [1,2].

3. PARABOLIC CASE:

Consider a nonlinear parabolic partial differential equation modeling convection-diffusion processes given in the form

$$u_t - \alpha u_{xx} + f(u,t,x)u_x = 0 \tag{3.1}$$

on a defined finite region on R with $t > 0$. Assume α is a constant, f is a smooth function of t,x,u, and the initial/boundary conditions are given. Of course, to obtain a quantitative solution, the specific form of f is required. When it is, any separable terms in x and t will be designated by $-g(x,t)$ and any remaining term dependent on u, and multiplying u_x can be written $N(u,u_x)$. Let $L_t = \partial/\partial t$ and $L_x = \partial^2/\partial x^2$ and write (2.1) as

$$L_t u - \alpha L_x u = g - N(u,u_x) \tag{3.2}$$

The decomposition method solves for each linear operator term in turn; thus

$$L_t u = g + \alpha L_x u - N(u,u_x)$$

$$L_x u = -\alpha^{-1} g + \alpha^{-1} L_t u + \alpha^{-1} N(u,u_x)$$

Since $L_t^{-1} L_t u = u - A = u - u(x,0)$ and $L_x^{-1} L_x u = u - B - Cx$, we obtain

$$u = A + L_t^{-1} g + \alpha L_t^{-1} L_x u - L_t^{-1} N(u,u_x)$$

$$u = B + Cx - \alpha^{-1} L_x^{-1} g + \alpha^{-1} L_x^{-1} L_t u + \alpha^{-1} L_x^{-1} N(u,u_x)$$

A is the initial condition, B and C are evaluated from the remaining two conditions. We add the two equations for u and divide by two, obtaining a single equation for u. If we define

$$\begin{aligned} u_0 &= (1/2)\{u(x,0) + B + Cx + L_t^{-1} g - \alpha^{-1} L_x^{-1} g\} \\ K &= -(1/2)\{\alpha L_t^{-1} L_x + \alpha^{-1} L_x^{-1} L_t\} \\ G &= -(1/2)\{L_t^{-1} - \alpha^{-1} L_x^{-1}\} \end{aligned} \tag{3.3}$$

we have

$$u = u_0 + Ku + G \cdot N(u,u_x)$$

The solution by decomposition is $u = \sum_{n=0}^{\infty} u_n$ with the given u_0 and the remaining components given by

$$u_{n+1} = Ku_n + GA_n \tag{3.4}$$

for $n \geq 0$, where the A_n are defined [1] for $N(u,u_x)$ with $N(u,u_x) = \sum_{n=0}^{\infty} A_n$. Thus

$$u = \sum_{n=0}^{\infty} K^n u_0 + G \cdot Nu \tag{3.5}$$

is the complete solution since the A_n are easily evaluated for any specific function f. A practical solution is given by $\phi_n = \sum_{i=0}^{n-1} u_i$ since the solution converges generally quite rapidly.

The solution can also be made by operating on (2.2) with $(L_t^{-1} - \alpha^{-1}L_x^{-1})$. Let us designate the solution of $L_t u = 0$ by ϕ_t and of $L_x u = 0$ by ϕ_x, i.e., $L_t^{-1}L_t u = u - \phi_t$ and $L_x^{-1}L_x u = u - \phi_x$. We obtain

$$u - \phi_t - \alpha L_t^{-1}L_x u - \alpha^{-1}L_x^{-1}L_t u + u - \phi_x$$

$$= (L_t^{-1} - \alpha^{-1}L_x^{-1})(g - N(u,u_x))$$

$$u = (1/2)(\phi_t + \phi_x) + (1/2)(L_t^{-1} - \alpha^{-1}L_x^{-1})g$$

$$- (1/2)(\alpha L_t^{-1}L_x + \alpha^{-1}L_x^{-1}L_t)u$$

$$-(1/2)(L_t^{-1} - \alpha^{-1}L_x^{-1})N(u,u_x)$$

or

$$u = u_0 + Ku + G \cdot N(u,u_x) \tag{3.6}$$

Though the result is the same and ϕ_t, ϕ_x are evaluated from the given conditions, writing u_0 as

$$u_0 = (1/2)(\phi_t + \phi_x) + (1/2)(L_t^{-1} - \alpha^{-1}L_x^{-1})g \tag{3.7}$$

means the result is not limited to the given parabolic equation - the derivatives can be of any order. The nonlinear term $N(u,u_x)$ can also be more general. The A_n can be generated for composite nonlinear functions [1].

4. ELLIPTIC CASE:

The elliptic equation $\nabla^2 u + k(x,y,z)u = f(x,y,z)$ arises in some problems of physics and engineering. Let's consider the case:

$$u_{xx} + u_{yy} + k(x,y)u = 0$$

$$k(x,y) = x^2 + y^2 \tag{4.1}$$

Define $L_x = \partial^2/\partial x^2$, $L_y = \partial^2/\partial y^2$ and write

$$[L_x + L_y]u + ku = 0$$

Solve for each linear operator term in turn. Then

$$\begin{aligned} L_x u &= - ku - L_y u \\ L_y u &= - ku - L_x u \end{aligned} \tag{4.2}$$

Apply the inverses L_x^{-1} to the first, L_y^{-1} to the second. Then since

$L_x^{-1} L_x u = u - \phi_x$, $L_y^{-1} L_y u = u - \phi_y$, where ϕ_x, ϕ_y are defined by the initial/boundary

conditions, we have

$$u = \phi_x - L_x^{-1}[x^2 + y^2]u \tag{4.3a}$$

$$u = \phi_y - L_y^{-1}[x^2 + y^2]u \tag{4.3b}$$

Choosing convenient conditions $u = 0$ at $x = 0$ or $y = 0$ and $u = \sin y$ at $x = 1$ and $\sin x$ at $y = 1$, we have

$$\phi_x = A + Bx = x \sin y$$

$$\phi_y = C + Dy = y \sin x$$

Let ϕ_x and ϕ_y represent the u_0 term of the decomposition $u = \sum_{n=0}^{\infty} u_n$ in the two equations for u in (4.3a,b). Thus, considering both in parallel

$$u_0 = x \sin y \tag{4.4a}$$

$$u_0 = y \sin x \tag{4.4b}$$

so that

$$u = u_0 - L_x^{-1}(x^2 + y^2) \sum_{n=0}^{\infty} u_n$$

$$u = u_0 - L_y^{-1}(x^2 + y^2) \sum_{n=0}^{\infty} u_n$$

Thus

$$u_{n+1} = -L_x^{-1}(x^2 + y^2)u_n \qquad (4.5a)$$

$$u_{n+1} = -L_y^{-1}(x^2 + y^2)u_n \qquad (4.5b)$$

for $n > 0$ where u_1 for (4.5a) uses (4.4a), etc. For example,

$$u_1 = - L_x^{-1}(x^2 + y^2)u_0 \qquad (4.6a)$$

$$u_1 = -L_y^{-1}(x^2 + y^2)u_0 \qquad (4.6b)$$

The one-term approximation to u is given by $\Phi_1 = (x \sin y + y \sin x)/2$. We see using the first terms of the trigonometric series that $\Phi_1 = xy$. We observe from the u_1 term and the second term of the expansion for sin y, we get $-x^3y^3/3!$.

From the u_2 term $-L_x^{-1}L_y[x^3/3!)\sin y]$ and the third term of sin y, we get $x^5y^5/5!$, etc. The n-term approximant Φ_n is given by the n-term series for sin xy plus noise terms. To save computation, we can substitute u = sin xy and verify it is indeed the correct solution. If we calculate several terms, it is easy to see cancellation of terms other than the series for sin xy. Or, if analytic nonlinearities f(u) are involved, the appropriate A_n polynomials are generated and we let $f(u) = \sum_{n=0}^{\infty} A_n$.

To make some checks of accuracy of the methodology, we consider the one-dimensional case $d^2u/dx^2 - 40xu = 2$ with $u(-1) = u(1) = 0$. Here $L = d^2/dx^2$ and we have $Lx = 2 + 40xu$. This is a relatively stiff case because of the large coefficient of u, and the nonzero forcing function yields an additional Airy-like function. Operating with L^{-1} yields $u = A + Bx + L^{-1}(2) + L^{-1}(40xu)$. Let $u_0 = A + Bx + L^{-1}(2) = A + Bx + x^2$ and let $u = \sum_{n=0}^{\infty} u_n$ with the components to be determined so that

the sum is u. We identify $u_{n+1} = L^{-1}(40xu_n)$. Then all components can be determined, e.g., $u_1 = (20/3)Ax^3 + (10/3)Bx^4 + 2x^5$ and $u_2 = (80/9)Ax^6 + (200/63)Bx^7 + (10/7)x^8$, etc. An n-term approximant $\phi_n = \sum_{i=0}^{n-1} u_i$ with n = 12 for x = 0.2 is given by -0.135649, for x = 0.4 is given by -0.113969, for x = 0.6 is given by -0.083321, for x = 0.8 is given by -0.050944, and for x = 1.0 is, of course, zero. These easily-obtained results are correct to seven digits. We see that a better solution is obtained and much more easily than by variational methods. The solution is found just as easily for nonlinear versions without linearization.

REFERENCES

1. ADOMIAN, G., Nonlinear Stochastic Operator Equations, Academic Press, (1986).
2. ADOMIAN, G., Applications of the Decomposition Method to the Navier-Stokes Equations, J. Math. Anal. and Applic., 119, 1 and 2, (1986).
3. ADOMIAN, G., Decomposition Solution for Duffing and Van der Pol Oscillators, Int. J. of Math. Sciences, 9, 4, (1986), 731-2.
4. BELLOMO, N. and RIGANTI, R., Semilinear Stochastic Systems: Analysis With the Method of the Stochastic Green's Function and Application in Mechanics, J. Math. Anal. and Applic. 96, 2, (1983).
5. BELLOMO, N. and MONACO R., A Comparison Between Decomposition Methods and Perturbation Techniques for Nonlinear Random Differential Equations, J. Math. Anal. adn Applic., 110, 2, (1985).
6. BONZANI, I., On a Class of Nonlinear Stochastic Dynamical Systems: Analysis of the Transient Behavior, J. Math. Anal. and Applic., in press.
7. RACH, R., A Convenient Computational Form for the A_n Polynomials, J. Math. Anal. and Applic., 102 2, (1984), 415-419.
8. GABETTA, E., On a Class of Semilinear Stochastic Systems in Mechanics with Quadratic Type Nonlinearities: Time Evolution of the Probability Density, J. Math. Anal. and Applic., in press.
9. RACH, R., On Continuous Approximate Solutions of Nonlinear Differential Equations J. Math. Anal. and Applic., in press.
10. RACH, R., On the Adomian (Decomposition) Method and Comparisons with Picard's Method, J. Math. Anal. and Applic., in press.
11. BIGI, D. and RIGANTI, R., Solutions of Boundary-Value Problems by Decompoistion Method.
12. BONZANI, I., Analysis of Stochastic Van der Pol Oscillators Using the Decomposition Method, Complex and Distributed Systems: Analysis, Simulation, and Control, eds. S. C. Tzafestas and P. Borne, Elsevier, (1986).
13. RIGANTI, R., Analytical Study of the Class of Nonlinear Stochastic Autonomous Oscillators with One Degree of Freedom, Meccanica, 14, 4, (1979), 180-86.
14. BELLOMO, N., and RIGANTI, R., Nonlinear Stochastic Systems in Physics and Mechanics, World Scientific Publishing Co., Singapore, (1987).
15. ADOMIAN, G., Applications of Nonlinear Stochastic Systems Theory to Physics, Reidel, Dordrecht, (1988).