A NOTE ON WEAKLY θ-CONTINUOUS FUNCTIONS

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ABSTRACT. Recently a new class of functions between topological spaces, called weakly θ-continuous functions, has been introduced and studied. In this paper we show how an appropriate change of topology on the domain of a weakly θ-continuous function reduces it to a weakly continuous function. This paper examines some of the consequences of this result.

PRELIMINARIES.

Throughout this paper X and Y denote topological spaces on which no separation axioms are assumed unless explicitly stated. A subset S of a space X is said to be *regular open* (resp. *regular closed*) if $\text{int}(\text{cl}S) = S$ (resp. $\text{cl}(\text{int}S) = S$), where $\text{cl}S$ and $\text{int}S$ denote the closure of S and the interior of S, respectively. Henceforth we denote $\text{int}(\text{cl}S)$ by the Bourbaki notation $\alpha(S)$. A point $x \in X$ is called a δ-*cluster* (resp. θ-*cluster*) *point* of S if $S \cap \alpha(U) \neq \phi$ (resp. $S \cap \text{cl}(U) \neq \phi$) for each open set U containing x. The set of all δ-cluster (resp. θ-cluster) points of S is called the δ-*closure* (resp. θ-*closure*) of S and is denoted by $\text{cl}_\delta(S)$ (resp. $\text{cl}_\theta(S)$). A subset S is said to be δ-*closed* (resp. θ-*closed*) if $\text{cl}_\delta(S) = S$ (resp. $\text{cl}_\theta(S) = S$).

A subset is called δ-*open* (resp. θ-*open*) if its complement is δ-closed (resp. θ-closed). The family of all δ-open (resp. θ-open) subsets of (X,T) forms a topology T_δ (resp. T_θ) on X.

Since the intersection of two regular open subsets of (X,T) is regular open, the family of all regular open subsets of (X,T) forms a base for a topology T_s on X,

called the *semi-regularization* of T.   It turns out that $T_s = T_\delta$, and that

$T_\theta \subset T_s \subset T$ for any topological space (X,T).   The relationship between T and

T_s is considered in [MRV].   In particular, we note that (X,T) is semi-regular if

and only if $T = T_s$.

A function $f : X \to Y$ is called *weakly continuous* (resp. *θ-continuous*) if for

each point $x \in X$ and each open neighbourhood V of $f(x)$, there is an open

neighbourhood U of x such that $f(U) \subset clV$ (resp. $f(clU) \subset clV$).   These concepts

were considered by Levine [L] and Fomin [F] respectively.

CHANGE OF TOPOLOGY

DEFINITION 1.   A function $f : (X,T) \to (Y,\mathcal{U})$ is called *weakly θ-continuous*

if for each point $x \in X$ and each open neighbourhood V of $f(x)$ there is an open

neighbourhood U of x such that $f(\alpha(U)) \subset clV$.

Cammaroto and Noiri [CN] point out with their Examples 3.2 and 3.3 that the

class of weakly θ-continuous functions lies strictly between the class of θ-continuous

functions and the class of weakly continuous functions.   In a recent paper [GMRV],

Gauld, Vamanamurthy and the present authors observed that there are many properties

closely related to the notion of continuity which in fact coincide with continuity if

the topology on either the domain or the range or both is changed.   They called such

a property a *continuity property* of functions.

PROPOSITION 1.   Weak θ-continuity is not a continuity property.

PROOF.   We refer to Lemma 2 of [GMRV].   It is shown in the proof of Theorem 2

of [GMRV] that f and g are θ-continuous, and hence they are weakly θ-continuous,

while h is not weakly continuous and hence not weakly θ-continuous.   So the result

follows from Lemma 2 of [GMRV].

We now show that a change of topology is germane to the discussion of weak

θ-continuity. We observe that if the domain of a weakly θ-continuous function is

retopologized appropriately, the function reduces to a weakly continuous function.

Baker [B2] has also made this observation. He used the term weakly δ-continuous in

place of weakly θ-continuous. The point should be made that weak continuity is not a

continuity property [GMRV, Theorem 2].   Recently, Noiri [N2] has studied another

class of functions, the weakly α-continuous functions, which also reduce to weakly

continuous functions when the domain is retopologized appropriately [N2, Lemma 2.2].

The proof of Theorem 1 follows directly from Definition 1.

THEOREM 1.   A function $f : (X,T) \to (Y,U)$ is weakly θ-continuous if and only if $f : (X,T_s) \to (Y,U)$ is weakly continuous.

Now most of the results of Cammaroto and Noiri [CN] can be reinterpreted in the light of Theorem 1.   For example, their Theorem 3.4 is an immediate consequence of Theorem 1,   their Theorem 3.5 (f) is a restatement of Theorem 1 of Levine [L], while Theorem 3.5(e) follows from Theorem 1(e) of Espelie and Joseph [EJ].   Other parts of Theorem 1 of [EJ] enable us to provide some further alternative characterizations of weak θ-continuity.

PROPOSITION 2.   For a function $f : (X,T) \to Y$ the following are equivalent:

(a) f is weakly θ-continuous.

(b)   For each $B \subset Y$, $T_s cl(f^{-1}(int(cl_\theta B))) \subset f^{-1}(cl_\theta B)$.

(c)   For each open set V in Y, $T_s cl(f^{-1}(\alpha(V))) \subset f^{-1}(clV)$.

(d)   For each regularly closed subset B of Y, $T_s cl(f^{-1}(intB)) \subset f^{-1}(B)$.

On the other hand, we can go in the opposite direction and use Theorem 3.7 of [CN] to obtain the following result for weak continuity.

PROPOSITION 3.   If $f : X \to Y$ is weakly continuous then

(a) $f^{-1}(F)$ is closed for each θ-closed   subset F of Y,

(b) $f^{-1}(V)$ is open for each θ-open subset V of Y.

The Example 3.9 of [CN] can be modified to show that the converse of Proposition 3 is false.

We observe that weak continuity is preserved under restriction of domain and under products. This observation together with Theorem 1 can be used to obtain the results of [CN, §4].   For example, Lemma 4 of [MRV] gives Theorem 4.1 of [CN],   while Lemma 3 of [MRV] provides a proof of Theorem 4.2 and 4.3 of [CN].

DEFINITION 2.   A function $f : (X,T) \to (Y,U)$ is called

(i)   *almost open* if $f : (X,T_s) \to (Y,U)$ is open,

(ii)   *δ-open* if $f : (X,T_s) \to (Y,U_s)$ is open,

(iii) *super open* if $f : (X,T) \to (Y,U_s)$ is open.

DEFINITION 3.   A subset S of (X,T) is called *N-closed* if for any open cover C of S there is a finite subcollection $\mathcal{D}$ of C such that $S \subset \cup \{\alpha(V) : V \in \mathcal{D}\}$.

Applying this change of topology technique to Lemma 2 of Rose [R] gives the following result.

PROPOSITION 4.   Let $f : X \to Y$ and $g : Y \to Z$.

(a)   If $g \circ f$ is weakly θ-continuous and f is an almost open surjection, then g is weakly continuous.

(b) If $g \circ f$ is weakly θ-continuous and f is an δ-open surjection, then g is weakly θ-continuous.

(c) If $g \circ f$ is weakly continuous and f is a super open surjection, then g is weakly θ-continuous.

PROPOSITION 5.   If $f : X \to Y$ is weakly θ-continuous and Y is Hausdorff, then for each N-closed subset C of X, $f(C)$ is δ-closed in Y.

PROOF.   Since $f : (X,T_s) \to (Y,U)$ is weakly continuous and $U_s \subset U$, we have $f : (X,T_s) \to (Y,U_s)$ is weakly continuous and hence subweakly continuous. If C is N-closed in (X,T) then C is compact in (X,T_s) so by Corollary 3 to Theorem 2 of Baker [B1], $f(C)$ is closed in (Y,U_s) and hence δ-closed in (Y,U).

Using a similar argument to that in the proof of Proposition 5, and Corollary 2 to Theorem 2 of Baker [B1], we obtain the following result.

PROPOSITION 6.   If $f : X \to Y$ is weakly θ-continuous and Y is Hausdorff, then for each N-closed subset C of Y, $f^{-1}(C)$ is δ-closed in X.

PROPOSITION 7.   Let Y be a Hausdorff space, $f : X \to Y$ be δ-continuous and $g : X \to Y$ be weakly θ-continuous. Then the set $\{x \in X : f(x) = g(x)\}$ is a δ-closed subset of X.

PROOF.   We have $f : (X,T_s) \to (Y,U_s)$ is continuous, $g : (X,T_s) \to (Y,U_s)$ is weakly continuous and (Y,U_s) is Hausdorff, and apply Theorem 6 of Baker [B1].

Finally we observe that Theorem 7.8 of [CN] is a corollary of Theorem 10 of [N1]. For, if $f : (X,T) \to (Y,U)$ is weakly θ-continuous then $f : (X,T) \to (Y,U_s)$ is weakly θ-continuous, so that $f : (X,T_s) \to (Y,U_s)$ is weakly continuous. Hence $G(f)$ is closed in $(X \times Y, T_s \times U_s)$, and thus δ-closed in $(X \times Y, T \times U)$.

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