Internat. J. Math. & Math. Sci.
VOL. 12 NO. 1 (1989) 119-122

ON BAICA'S TRIGONOMETRIC IDENTITY

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(Received May 21, 1987 and in revised form September 7, 1987)

ABSTRACT: The author obtains three new elementary trigonometric identities including

$$\left(\frac{\sin p\theta}{\sin \theta}\right)^{p-2} = 2^{\frac{(p-1)(p-2)}{2}} \prod_{k=1}^{p-1} \left(\cos 2\theta - \cos \frac{2\pi k}{p}\right)^{p-1-k},$$

which, letting θ approach 0, gives a simple derivation of a recent result of M. Baica [1],

$$p^{p-2} = 2^{\frac{(p-1)(p-2)}{2}} \prod_{k=1}^{p-2} \left(1 - \cos \frac{2\pi k}{p}\right)^{p-1-k}.$$

KEY WORDS AND PHRASES. Trigonometric identities, Chebyshev polynomials.
1980 AMS SUBJECT CLASSIFICATION CODE. 12E12, 42A05.

1. INTRODUCTION

Some interesting combinatorial relations that are difficult to prove directly can be easily derived as limiting cases of simpler relations. We here show that the interesting recent result of Baica [1],

$$p^{p-2} = 2^{\frac{(p-1)(p-2)}{2}} \prod_{k=1}^{p-2} \left(1 - \cos \frac{2\pi k}{p}\right)^{p-1-k}, \tag{1.1}$$

for $p > 3$, can be proved this way, by proving the elementary relations (1.2) and (1.3) below. Then (1.1) follows from (1.3) by letting θ approach 0 on both sides of (1.3). (Since the $p-1$ term of (1.3) is 1, the product in (1.3) can be considered as having only $p-2$ terms.)

The identities

$$\left(\frac{\sin p\theta}{\sin \theta}\right)^{2} = 2^{p-1} \prod_{k=1}^{p-1} \left(\cos 2\theta - \cos \frac{2\pi k}{p}\right) \tag{1.2}$$

and

$$\left(\frac{\sin p\theta}{\sin \theta}\right)^{p-2} = 2^{\frac{(p-1)(p-2)}{2}} \prod_{k=1}^{p-1} \left(\cos 2\theta - \cos \frac{2\pi k}{p}\right)^{p-1-k}, \tag{1.3}$$

for all $p > 2$, can be easily proved in several ways.

Proofs using polynomials seem simplest and because Chebyshev polynomials (of the second kind), $U_{p-1}(x) = \sin(p \cos^{-1}(x))/\sin(\cos^{-1}(x))$, are well known, they will be used in the arguments here.

The only properties of $U_{p-1}(x)$ used here are the facts that $U_{p-1}(x)$ is a polynomial of degree $p-1$ with leading coefficient 2^{p-1} and zeros $\cos(\pi k/p)$, $k = 1,\dots,p-1$, in $(-1,1)$. For completeness we will prove these facts in the next section.

2. CHEBYSHEV POLYNOMIALS.

The Chebyshev polynomials of the first kind are defined by $T_p(x) = \cos(p\cos^{-1}(x))$. If $\theta = \cos^{-1}(x)$, then $T_p(x) + i\,U_{p-1}(x)\sqrt{1-x^2} = \cos p\theta + i\sin p\theta = (\cos\theta + i\sin\theta)^p = (x + i\sqrt{1-x^2})^p$. Expanding this last expression shows $T_p(x)$ and $U_{p-1}(x)$ are polynomials. Inductively assume $T_p(x)$ and $U_{p-1}(x)$ both have leading coefficients 2^{p-1}, this being clear for $p = 1$. By expanding $(x + i\sqrt{1-x^2})^p(x + i\sqrt{1-x^2})$ it is seen that $T_{p+1}(x)$ and $U_p(x)$ have leading coefficients 2^p. Finally, that the roots of the $p-1$ degree polynomial $U_{p-1}(x)$ are $\cos\frac{\pi k}{p}$, $k = 1,\dots,p-1$, can be directly verified in the definition of $U_{p-1}(x)$ given in the introduction.

3. PROOFS OF (1.2) AND (1.3).

Writing the polynomial $U^2_{p-1}(x)$ in terms of its roots, $U^2_{p-1}(x) = \left(2^{p-1}\prod_{k=1}^{p-1}\left(x - \cos\frac{\pi k}{p}\right)\right)^2$, and by pairing terms in increasing and decreasing order, $U^2_{p-1}(x) = 2^{2(p-1)}\prod_{k=1}^{p-1}\left(x - \cos\frac{\pi k}{p}\right)\left(x - \cos\frac{\pi(p-k)}{p}\right)$. It follows from $\cos\frac{\pi(p-k)}{p} = -\cos\frac{\pi k}{p}$ and a double angle formula that

$$U^2_{p-1}(x) = 2^{2(p-1)}\prod_{k=1}^{p-1}\left(x^2 - \cos^2\left(\frac{\pi k}{p}\right)\right)$$

$$= 2^{2(p-1)}\prod_{k=1}^{p-1}\left(2x^2 - 1 - \cos\frac{2\pi k}{p}\right)/2 \tag{3.1}$$

Now let $x = \cos\theta$ to see that

$$\left(\frac{\sin p\theta}{\sin\theta}\right)^2 = 2^{2(p-1)}\prod_{k=1}^{p-1}\left(\frac{\cos 2\theta - \cos\frac{2\pi k}{p}}{2}\right)$$

proving (1.2).

Now denote $2x^2-1$ by $T_2(x)$ in equation (3.1) and raise both sides of (3.1) to the power $p-2$:

$$(U_{p-1}(x))^{2(p-2)} = 2^{(p-1)(p-2)}\left(\prod_{k=1}^{p-1}\left(T_2(x) - \cos\frac{2\pi k}{p}\right)\right)^{p-2}.$$

Rewriting the product on the right in ascending and descending order and noting that $\cos\frac{2\pi k}{p} = \cos\frac{2\pi(p-k)}{p}$ gives

$$(U_{p-1}(x))^{2(p-2)}$$

$$= 2^{(p-1)(p-2)} \prod_{k=1}^{p-1} \left(T_2(x) - \cos \frac{2\pi k}{p}\right)^{p-1-k} \cdot \prod_{k=1}^{p-1} \left(T_2(x) - \cos \frac{2\pi(p-k)}{p}\right)^{p-1-k}$$

$$= 2^{(p-1)(p-2)} \left(\prod_{k=1}^{p-1} \left(T_2(x) - \cos \frac{2\pi k}{p}\right)^{p-1-k}\right)^2.$$

Thus

$$\pm(U_{p-1}(x))^{p-2} = 2^{\frac{(p-1)(p-2)}{2}} \prod_{k=1}^{p-1} \left(T_2(x) - \cos \frac{2\pi k}{p}\right)^{p-1-k}.$$

Since this is a polynomial identity, the algebraic sign can be determined by evaluating it at any x which gives non-zero values. Let x approach 1. Then $\lim_{x \to 1} U_{p-1}(x) = \lim_{\theta \to 0} \frac{\sin p\theta}{\sin \theta} = p$, and $T_2(1) = 1$, so the positive sign must clearly be chosen. Taking $x = \cos \theta$ proves (1.3). As noted earlier, equation (1.1) is the limiting case of (1.3) at $\theta = 0$.

4. FURTHER CONSEQUENCES OF (1.2).

The limiting case at $\theta = 0$ of (1.2) is

$$p^2 = 2^{p-1} \prod_{k=1}^{p-1} \left(1 - \cos \frac{2\pi k}{p}\right).$$

Now

$$1 - \frac{\sin^2 \theta}{\sin^2 \frac{\pi k}{p}} = 1 - \frac{(1 - \cos 2\theta)/2}{\left(1 - \cos \frac{2\pi k}{p}\right)/2} = \frac{\cos 2\theta - \cos \frac{2\pi k}{p}}{1 - \cos \frac{2\pi k}{p}},$$

so, by (1.2) and the above limiting case of (1.2),

$$\left(\frac{\sin p\theta}{\sin \theta}\right)^2 = p^2 \prod_{k=1}^{p-1} \left(1 - \frac{\sin^2 \theta}{\sin^2 \frac{\pi k}{p}}\right). \tag{4.1}$$

For odd p this is the square of the known identity (see Spiegel [2]) that for odd p

$$\frac{\sin p\theta}{\sin \theta} = p \prod_{k=1}^{\frac{p-1}{2}} \left(1 - \frac{\sin^2 \theta}{\sin^2 \frac{\pi k}{p}}\right), \tag{4.2}$$

thus generalizing (4.2) to all p. When p is even, the $p/2$ term in (4.1) is $1 - \sin^2 \theta = \cos^2 \theta$, so pleasing versions of (4.2) when p is even would be

$$\frac{\sin p\theta}{\sin \theta} = p \cos \theta \prod_{k=1}^{[\frac{p-1}{2}]} \left(1 - \frac{\sin^2 \theta}{\sin^2 \frac{\pi k}{p}}\right) \tag{4.3}$$

or

$$\frac{\sin p\theta}{\sin 2\theta} = \frac{p}{2} \prod_{k=1}^{[\frac{p-1}{2}]} \left(1 - \frac{\sin^2 \theta}{\sin^2 \frac{\pi k}{p}}\right). \tag{4.4}$$

Finally, in view of the identity

$$2 \sin i \sin j = \cos(i-j) - \cos(i+j),$$

$$\prod_{k=1}^{p-1}\left(\cos 2\theta - \cos \frac{2\pi k}{p}\right) = \prod_{k=1}^{p-1}\left(\cos\left(\frac{2\pi k}{p} - \pi\right) - \cos(2\theta+\pi)\right)$$

$$= \prod_{k=1}^{p-1}\left(2 \sin\left(\theta + \frac{\pi k}{p}\right) \sin\left(\theta + \frac{\pi(p-k)}{p}\right)\right)$$

$$= 2^{p-1}\left(\prod_{k=1}^{p-1} \sin\left(\theta + \frac{\pi k}{p}\right)\right)^2 .$$

Thus by equation (1.2) we see that the known identity

$$\frac{\sin p\theta}{\sin \theta} = 2^{p-1} \prod_{k=1}^{p-1} \sin\left(\theta + \frac{\pi k}{p}\right) \tag{4.5}$$

can be derived as well (Carlson [3]), or, conversely, can be used as the starting point to prove (1.2) and (1.3).

We could not find a reference to an elementary proof of (4.5). Many texts use the gamma function (as in Carlson [3]) to derive the general formula and others only prove the limiting case $\theta = 0$ (for example, see Spiegel [2]).

We conclude with the following direct proof of identity (4.5). By Euler's relation, $\sin p\theta = \left(e^{ip\theta} - e^{-ip\theta}\right)/2i$, so

$$\prod_{k=0}^{p-1} \sin\left(\theta + \frac{\pi k}{p}\right) = \prod_{k=0}^{p-1} \frac{e^{i\left(\theta + \frac{\pi k}{p}\right)} - e^{-i\left(\theta + \frac{\pi k}{p}\right)}}{2i}$$

$$= \prod_{k=0}^{p-1} \left[\left(e^{2i\theta} - e^{-i\frac{2\pi k}{p}}\right) \frac{e^{-i\theta}\, e^{i\frac{\pi k}{p}}}{2i}\right] .$$

Note that, if $z = e^{2i\theta}$, the conjugates, $\bar{z}_k = e^{-i\frac{2\pi k}{p}}$, of the ordinary roots of z^p-1 are again all the roots of z^p-1. Thus part of the above is a factorization of z^p-1. Computing the product gives

$$\prod_{k=0}^{p-1} \sin\left(\theta + \frac{\pi k}{p}\right) = \left[\left(e^{2i\theta}\right)^p-1\right] \frac{e^{-ip\theta}\left(e^{\frac{i\pi}{p}}\right)^{\frac{p(p-1)}{2}}}{(2i)^p}$$

$$= \frac{\left(e^{ip\theta} - e^{-ip\theta}\right)i^{p-1}}{(2i)^p} = \frac{\sin p\theta}{2^{p-1}} .$$

This proof of (4.5) is due to Professor James A. Wilson. I thank him for allowing me to include it here.

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