Internat. J. Math. & Math. Sci.
VOL. 12 NO. 1 (1989) 205-207

ON ANTI-COMMUTATIVE SEMIRINGS

J.S. RATTI and Y.F. LIN

Department of Mathematics
University of South Florida
Tampa, Florida 33620

(Received March 28, 1988)

ABSTRACT. An anticommutative semiring is completely characterized by the types of multiplications that are permitted. It is shown that a semiring is anticommutative if and only if it is a product of two semirings R_1 and R_2 such that R_1 is left multiplicative and R_2 is right multiplicative.

KEY WORDS AND PHRASES. Semiring, anticommutative, isomorphism.
1980 AMS SUBJECT CLASSIFICATION CODES. 16A78

A semiring is a non-empty set R equipped with two binary operations, called addition + and multiplication (denoted by juxtaposition), such that R is multiplicatively a semigroup, additively a commutative semigroup and multiplication is distributive across the addition both from the left and the right.

A semiring R is called anti-commutative if and only if for arbitrary $x, y \in R$ the relation $x \neq y$ always implies $xy \neq yx$.

Let R_1 and R_2 be semirings, then $R_1 \times R_2$ is the semiring with the following operations:

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$

$$(x_1, x_2) \cdot (y_1, y_2) = (x_1y_1, x_2y_2).$$

Suppose R is a commutative semigroup under +, and if we define multiplication in R of type

(T_1) $xy = x$ for all $x, y \in R$

or

(T_2) $xy = y$ for all $x, y \in R$,

then it is easily seen that R is an anti-commutative semiring.

A natural question that arises is the following: Suppose R is an anti-commutative semiring. Does the multiplication in R have to be of type (T_1) or (T_2)? to answer this question, we prove the following:

THEOREM 1. A semiring R is anti-commutative if and only if R is isomorphic to $R_1 \times R_2$, where R_1 is a semiring with multiplication of type (T_1) and R_2 is a seniring with multiplication of type (T_2).

We shall need the following lemma, whose proof is contained in [1,p.75], to prove Theorem 1.

LEMMA. Let R be an anti-commutative semiring, then for arbitrary x, y, z, $\in$ R we have

(i) $x^2 = x$

(ii) $xyz = xz$

PROOF OF THEOREM 1. Since R is non empty, let $a \in R$. Set $R_1 = Ra$ and $R_2 = aR$. By using the lemma, it is obvious that Ra and aR are semirings and multiplication in Ra is of type (T_1) and multiplication in aR is of type (T_2).

Let $f: R \to Ra \times aR$, such that for each $x \in R$,

$$f(x) = (xa, ax).$$

then for $y \in R$, $f(y) = (ya, ay)$.

$$f(x+y) = ((x+y)a,\ a(x+y)) = (xa + ya,\ ax + ay)$$
$$= (xa, ax) + (ya, ay)$$
$$= f(x) + f(y).$$
$$f(xy) = (xya,\ axy)$$
$$= (xaya,\ axay) \quad \text{[By part (ii) of the Lemma]}$$
$$= f(x)f(y).$$

Thus, f is a homomorphism.

To show f is an isomorphism, let us define g: Ra x aR → R, such that g(xa,ay) = xy.

Then

$$(gof)(x) = g(f(x)) = g(xa,ax) = xa^2x = x^2 = x,$$

and

$$(fog)(xa,ay) = f[g(xa,ay)] = f(xy) = (xya,axy) = (xa,ay).$$

This shows that f is an isomorphism.

The proof for the converse is left to the reader.

THEOREM 2. Let R be an anti-commutative semiring. Then for an arbitrary x∈R, x + x = x.

PROOF: As in the proof of Theorem 1, we have

$$x = g(xa,ax).$$

Thus,

$$\begin{aligned} x + x &= g(xa + xa,\ ax + ax) \\ &= g(x^2a + x^2a,\ ax^2 + ax^2) \\ &= g(x(x + x)a,\ a(x + x)x) \\ &= g(xa,\ ax) \\ &= x. \end{aligned}$$

REFERENCES

1. LJAPIN, E.S. Semigroups, American Math Society Translation Providence, Rhode Island (1963)