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CONVOLUTION AND DIFFERENTIAL SUBORDINATION

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ABSTRACT. Let A denote the class of all functions f analytic in the open unit disc U with $f(0) = 0 = f'(0)-1$. Let h be any convex univalent analytic function on U with $h(0) = 1$ and $\text{Re } h(z) > 0$ in U. Let $g \in A$ be fixed. Denote by $S_g(h)$ the class of all functions $f \in A$ such that, $g*f(z) \neq 0$ in U and $\frac{a(g*f)'(z)}{(g*f)(z)} \prec h(z)$, $z \in U$ ($\prec$ denote subordination). It is proved in this paper that the class $S_g(h)$ is closed under convolution with convex functions. It has also been established that $S_g(h) \subseteq S_{\Phi * g}(h)$ where Φ is any convex univalent function in A. Four other classes are also defined and studied using mainly the convex hull method and the methd of differential subordination.

KEY WORDS AND PHRASES. Convolution, Convex function, Close-to-convex function, Quasi-convex function, Starlike function, Subordination.
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1. INTRODUCTION.

Let $U = \{z: |z| < 1\}$ and H(U) be the class of all holomorphic functions defined on U. Let $A = \{f \in H(U)/f(0) = 0, f'(0) = 1\}$. Let $f, g \in H(U)$ and $f(z) = \sum_0^\infty b_n z^n$. Then the convolution of f and g is denoted as

$$(f*g)(z) = \sum_0^\infty a_n b_n z^n.$$

Let g and $G \in H(U)$, g(z) is said to be <u>subordinate</u> to G(z) (written $g(z) \prec G(Z)$) in U if G(z) is univalent in U, $g(0) = G(0)$ and $g(U) \subseteq G(U)$. Let S*, K, Q and C denote the subclass of A consisting of Starlike univalent, convex univalent, Quasi-Convex and close-to-convex functions respectively. Let M_α denote the class of functions in A which are α-convex (Mocanu sense) in U.

As a general reference for the definitions and properties of the above classes the reader may consult the book by A.W. Goodman [1].

The aim of this paper is to give various subclasses of A, analogous to the classes S*, K, Q, C and M_α, and to study some properties of the new classes.

2. BASIC THEOREMS AND DEFINITION OF THE NEW CLASSES.

We need the following theorems to prove our main results.

THEOREM 2.1 [2]: Let $\phi \epsilon A$ be convex univalent, $g \epsilon S^*$ and $F \epsilon H(U)$ such that $\operatorname{Re} F(z) > 0$ for $\forall\, z \epsilon U$. Then $\frac{\phi * Fg}{\phi * g}$ lies in the convex hull of F(U).

THEOREM 2.2 [3]: Let $\beta, \nu \epsilon C$, $h \epsilon H(U)$ be convex univalent in U with $h(0) = 1$ and $\operatorname{Re}(\beta h(z) + \nu) > 0$, $\forall\, z \epsilon U$ and let

$$p(z) = 1 + p_1 z + \dots \epsilon H(U). \quad \text{Then}$$

$$p(z) + \frac{zp'(z)}{\beta p(z)+\nu} \prec h(z) \Rightarrow p(z) \prec h(z)$$

A modification of Theorem 2.2 is given in

THEOREM 2.3 [4]: Let $\beta, \nu \epsilon C$, $h \epsilon H(U)$ be convex univalent in U with $h(0) = 1$ and $\operatorname{Re}(\beta h(z) + \nu) > 0$, $\forall\, z \epsilon U$ and let $q \epsilon H(U)$ with $q(0) = 1$ and $q(z) \prec h(z)$, $\forall\, z \epsilon U$. If

$$p(z) = 1 + p_1 z + \dots \quad \text{is in } H(U), \text{ then}$$

$$p(z) + \frac{zp'(z)}{\beta q(z)+\nu} \prec h(z) \Rightarrow p(z) \prec h(z).$$

To avoid repetition we say once and for all in this paper, unless otherwise specified, g will denote a fixed function in A and h will always denote a convex univalent function on U with $h(0) = 1$ and $\operatorname{Re} h(z) > 0$ for $\forall\, z \epsilon U$.

DEFINITION 2.1: Let $S_g(h)$ denote the class of all functions $f \epsilon A$ such that $\frac{(g*f)(z)}{z} \neq 0$ in U and satisfying

$$\frac{z(g*f)'(z)}{(g*f)(z)} \prec h(z) \tag{2.1}$$

DEFINITION 2.2: Let $K_g(h)$ denote the class of all functions $f \epsilon A$ such that $(g*f)'(z) \neq 0$ in U and satisfying

$$1 + \frac{Z(g*f)''(z)}{(g*f)'(z)} \prec h(z) \tag{2.2}$$

DEFINITION 2.3: Denote by $C_g(h)$ the class of all functions $f \epsilon A$ such that $\frac{(g*f)(z)}{z} \neq 0$ for $\forall\, z \epsilon U$ and satisfying

$$\frac{z(g*f)'(z)}{(g*\Psi)(z)} \prec h(z), \ \forall\, z \epsilon U, \text{ for some } \Psi \epsilon S_g(h) \tag{2.3}$$

REMARK 2.1: If $g(z) = z/(1-z)^a$ (a real) then $S_g(h)$ and $C_g(h)$ coincides with the classes $S_a(h)$ and $C_a(h)$ respectively introduced in [4]. $K_g(h)$ is the class $K_a(h)$ introduced in [5]. For the choice $h(z) = \frac{1+z}{1-z}$, the classes $S_a(h)$ and $K_a(h)$ were investigated by S. Owa et al [6]. If $g(z) + \frac{z}{(1-z)^{2(1-\alpha)}}$ and $h(z) = \frac{1+(1-2\alpha)z}{1-z}$, $0 < \alpha < 1$, then $S_g(h)$ is nothing but the class of pre-Starlike functions of order α introduced by Ruschewegh in [7].

If $g(z)$ is taken as z then $S_g(h)$ is the class A and hence in general functions in $S_g(h)$ need not be univalent.

DEFINITION 2.4: Let α be any real number. Let $K_g^{\alpha}(h)$ denote the class of functions $f\epsilon A$ such that $\frac{(g*f)(z)}{z} \neq 0$ and $(g*f)'(z) \neq 0$ in U and

$$J_g(\alpha;\ f(z)) = \alpha(1 + \frac{z(g*f)''(z)}{(g*f)'(z)}) + (1-\alpha)\ \frac{z(g*f)'(z)}{(g*f)(z)} \tag{2.4}$$

is subordinate to $h(z)$.

REMARK 2.2: When $g(z) = z/(1-z)^a$ $K_g^{\alpha}(h)$ is the same as $K_a^{\alpha}(h)$ introduced in [5]. As it can be seen clearly that $K_g^{0}(h) = S_g(h)$ and $K_g^{'}(h) = K_g(h)$, $K_g^{\alpha}(h)$ provides a 'continuous passage' from the class $K_g(h)$ to $S_g(h)$ as α decreases from 1 to 0.

DEFINITION 2.5: Let $Q_g(h)$ denote the class of all functions $f\epsilon A$ such that $\frac{(g*f)(z)}{z} \neq 0$ in U and satisfying for some $\Phi\epsilon K_g(h)$

$$\frac{[z(g*f)'(z)]'}{(g*\Phi)'(z)} < h(z), \quad \forall\ z\epsilon U. \tag{2.5}$$

REMARK 2.3: When $g(z) = z/(1-z)^a$ we shall denote the class $Q_g(h)$ by $Q_a(h)$. If further a=1 and $h(z) = \frac{1+z}{1-z}$, $Q_a(h)$ is the class of Quasi-Convex functions introduced by K.I. Noor and D.K. Thomas [8].

3. MAIN THEOREMS.

THEOREM 3.1: If $f\epsilon S^*(h)$ and g is a convex function then $f\epsilon S_g(h)$.

PROOF: We have

$$\frac{z(g*f)'(z)}{(g*f)(z)} = \frac{(g*zf')(z)}{(g*f)(z)} = \frac{(g*\frac{zf'}{f}.f)(z)}{(g*f)\ (z)}$$

Since $f\epsilon S^*(h)$, $\mathrm{Re}(\frac{zf'(z)}{f(z)}) > 0$ and g convex, and hence by an application of Theorem A we get,

$$\frac{z(g*f)'(z)}{(g*f)(z)} < h(z)$$

which implies $f\epsilon S_g(h)$.

It is a well known fact that $K \subseteq S^*$ and $f \varepsilon K$ if and only $zf' \varepsilon S^*$, we shall now extend this fact to the class $S_g(h)$ and $K_g(h)$.

THEOREM 3.2: (i) $K_g(h) \subseteq S_g(h)$

(ii) $f \varepsilon K_g(h)$ if and only if $zf' \varepsilon S_g(h)$

PROOF: (i) Let $p(z) = \frac{z(g*f)'(z)}{(g*f)(z)}$, Logarithamic derivative of $p(z)$ and a multiplication by z gives

$$p(z) + \frac{zp'(z)}{p(z)} = 1 + \frac{z(g*f)''(z)}{(g*f)(z)} . \tag{3.1}$$

If $f \varepsilon K_g(h)$ then by Definition 2.2 the right side of (3.1), and hence the left side is subordinate, to $h(z)$. Applying Theorem 2.2 we get, $p(z) < h(z)$ and hence the result (i). To prove (ii) we have for any two functions f, $g \varepsilon A$ such that $\frac{(f*g)(z)}{z} \neq 0$, $(f*g)'(z) \neq 0$.

$$\frac{z(g*zf')'(z)}{(g*zf')*(z)} = 1 + \frac{z(g*f)''(z)}{(g*f)'(z)} . \tag{3.2}$$

Now, if $f \varepsilon K_g(h)$ then from (2.2) and (3.2) $\frac{z(g*zf')''(z)}{(g*zf')(z)} < h(z)$. Therefore $zf' \varepsilon S_g(h)$. Conversly, if $zf' \varepsilon S_g(h)$, from (2.1) and (3.2),

$1 + \frac{z(g*f)''(z)}{(g*f)'(z)} < h(z)$ and hence $f \varepsilon K_g(h)$.

REMARK 3.1: For $g(z) = z/(1-z)^a$ Theorem 3.1 gives Theorem 3 in [5] as a particular case.

Next we prove the classes $S_g(h)$ and $K_g(h)$ are closed under convolutions with convex univalent functions.

THEOREM 3.3: Let $\Phi \varepsilon A$ be convex univalent then for every $f \varepsilon S_g(h)$, $\Phi * f \varepsilon S_g(h)$.

PROOF: Let $F(z) = \frac{z(g*f)'(z)}{(g*f)(z)}$. If $f \varepsilon S_g(h)$, then $F < h$. Now,

$$\frac{z(g*\Phi*f)'(z)}{(g*\Phi*f)(z)} = \frac{(\Phi*z(g*f)')(z)}{(\Phi*(g*f))(z)} = \frac{(\Phi*F(g*f))(z)}{(\Phi*(g*f))(z)} .$$

Since $f \varepsilon S_g(h)$, $g*f \varepsilon S^*(h) \subseteq S^*$ and it follows from Theorem 2.1 that

$\frac{(\Phi*F(g*f))(z)}{(\Phi*(g*f))(z)}$ lies in the convex hull of $F(U)$. But $F < h$ and h is convex. Therefore the convex hull of $F(U)$ is a subset of $h(U)$ and the conclusion follows.

COROLLARY 3.1: Let $\phi \varepsilon A$ be convex univalent then for every $f \varepsilon K_g(h)$, $\Phi * f \varepsilon K_g(h)$.

PROOF: This follows easily from Theorem 3.2 (ii) and Theorem 3.3.

THEOREM 3.4: $S_g(h) \subseteq S_{\Phi * g}(h)$ for every convex univalent function ϕ with $\phi(0) = \phi'(0) = 1$.

PROOF: Let $f \in S_g(h)$, then by Theorem 3.3 $\Phi * f \in S_g(h)$. Hence

$\frac{z(g*\Phi*f)'(z)}{(g*\Phi*f)(z)} \prec h(z)$. That is $f \in S_{\Phi * g}(h)$. Normalizing condition on ϕ are forced because of the conditions on g's.

COROLLARY 3.2: $K_g(h) \subseteq K_{\Phi * g}(h)$ for every convex univalent function ϕ in A.

PROOF: Follows easily from Theorem 3.2 (ii) and Theorem 3.4.

REMARK 3.2: If $g(z) = z/(1-z)^{a+1} = K_{a+1}(z)$ and observing that $(h_\gamma * K_{a+1}) = K_a(z)$, where $h_\gamma(z) = \sum_1^\infty \frac{\gamma+1}{n+\gamma} z^n$ with $\gamma = a-1$. Then Theorem 3.4 and Corollary 3.2 reduce to the fact that $S_{a+1}(h) \subseteq S_a(h)$ and $K_{a+1}(h) \subseteq K_a(h)$ for $a \geq 1$. These two containment results are proved respectively in [4] and [5].

It follows easily from Definition 2.3 by taking $\Psi = f$ that $S_g(h) \subseteq C_g(h)$. We now prove that the class $C_g(h)$ is closed under convolution with a convex function.

THEOREM 3.5: Let $f \in C_g(h)$ with respect to a function $f_1 \in S_g(h)$. Then for every convex univalent function $\Phi \in A$, $\Phi * f \in C_g(h)$ with respect to $\Phi * f_1 \in S_g(h)$.

PROOF: It is clear from Theorem 3.3 that $\Phi * f_1 \in S_g(h)$. Let $F(z) = \frac{z(g*f)'(z)}{(g*f_1)(z)}$.

Since $f \in C_g(h)$ with respect to $f_1 \in S_g(h)$, it follows that $F(z) \prec h(z)$, $\forall\ z \in U$, in particular Re $F(z) > 0$, also we have $g*f_1 \in S^*$. Now

$$\frac{z(g*\Phi*f)'(z)}{(g*\Phi*f_1)(z)} = \frac{(\Phi*z(g*f)')(z)}{(\Phi*(g*f_1))\ (z)} = \frac{(\Phi*F(g*f_1))(z)}{(\Phi*(g*f_1))\ (z)}.$$

Applying Theorem 2.1 we get Theorem 3.5.

REMARK 3.3: If $g(z) = z/(1-z)^a$, and $\phi = h_\gamma(z)$, we get Theorem 4 of [4] as a particular case of Theorem 3.5.

THEOREM 3.6: $C_g(h) \subseteq C_{\Phi * g}(h)$ for every convex univalent function $\Phi \in A$.

PROOF: It follows exactly in the same way as Theorem 3.4 and is hence omitted.

REMARK 3.4: If $g(z) = z/(1-z)^a$ and $\Phi(z) = h_\gamma(z)$ with $\gamma = a-1$ we get Theorem 3 of [4].

THEOREM 3.7: (i) Let $\alpha > 0$, then $K_g^\alpha(h) \subseteq S_g(h)$

(ii) for $\alpha > \beta \geq 0$, $K_g^\alpha(h) \subseteq K_g^\beta(h)$.

PROOF: (i) Let $p(z) = \frac{z(g*f)'(z)}{(g*f)(z)}$, then $J_g(\alpha; f(z)) = p(z) + \frac{\alpha z p'(z)}{p(z)}$. Since $f \varepsilon K_g^{\alpha}(h)$ it follows that $J_g(\alpha; f(z)) < h(z)$. Now an application of Theorem 2.2 gives that $p(z) \prec h(z)$. Hence $f \varepsilon S_g(h)$.

(ii) The case $\beta = 0$ is contained in (i) so assume $\beta > 0$. Now,

$$J_g(\beta; f(z)) = (1-\beta)\frac{z(g*f)'(z)}{(g*f)(z)} + \beta\left(1 + \frac{z(g*f)''(z)}{(g*f)'(z)}\right)$$

$$= \left(1 - \frac{\beta}{\alpha}\right)\frac{z(g*f)'(z)}{(g*f)(z)} + \frac{\beta}{\alpha} J_g(\alpha; f(z))$$

by (i) $\frac{z(g*f)'(z)}{(g*f)(z)} \prec h(z)$ and by assumption $J_g(\alpha; f(z)) \prec h(z)$ and hence $J_g(\beta; f(z)) \prec h(z)$ (as $\beta/\alpha < 1$). Therefore $f \varepsilon K_g^{\beta}(h)$.

REMARK 3.5: If $g(z) = z/(1-z)$ and $h(z) = \frac{1+z}{1-z}$ the first part of Theorem 3.7 reduces to the result due to Mocanu and Reade [9] that all α-convex functions are starlike univalent and the second part of Theorem 3.7 reduces to a result of Sakaguchi [10]. If $g(z) = z/(1-z)^a$ then Theorem 3.7 gives Theorem 6 of Padmanabhan and Manjini [5] as a particular case.

THEOREM 3.8: (i) $K_g(h) \subseteq Q_g(h) \subseteq C_g(h)$

(ii) $f \varepsilon Q_g(h)$ if and only if $zf' \varepsilon C_g(h)$.

PROOF: (i) By taking $f = \Phi$ it follows easily from the definition of the class $Q_g(h)$ that $K_g(h) \subseteq Q_g(h)$. To prove the other inclusion, set

$$p(z) = \frac{z(g*f)'(z)}{(g*\Phi)(z)}. \quad \text{Then}$$

$$p(z) + \frac{zp'(z)}{\frac{z(g*\Phi)'(z)}{(g*\Phi)(z)}} = \frac{[z(g*f)'(z)]'}{(g*\Phi)'(z)} \tag{3.3}$$

If $f \varepsilon Q_g(h)$ then there exists a $\Phi \varepsilon K_g(h)$ such that the right hand side of (3.3) and hence the left hand side of (3.3) is subordinate to $h(z)$.

Since $\Phi \varepsilon K_g(h) \subseteq S_g(h)$ we have $\operatorname{Re} \frac{z(g*\Phi)'(z)}{(g*\Phi)(z)} > 0$ in U. Hence applying Theorem 2.3 we get $p(z) \prec h(z)$. That is $f \varepsilon C_g(h)$.

To prove (ii) we have for any two functions f and Φ satisfying the non-zero convolution conditions that

$$\frac{z[z(g*f)'(z)]'}{z(g*\Phi)'(z)} = \frac{z(g*zf')'(z)}{(g*z\Phi')(z)}. \tag{3.4}$$

Now, if $f \varepsilon Q_g(h)$ with respect to a function $\Phi \varepsilon K_g(h)$, then the left hand side of 3.4 is subordinate to $h(z)$. Now $z\Phi' \varepsilon S_g(h)$ by Theorem 3.2 (ii) and hence by the definition of $C_g(h)$ and by (3.4) $zf' \varepsilon C_g(h)$. Conversely, if $zf' \varepsilon C_g(h)$, then there exists a function $\Phi_1 \varepsilon S_g(h)$ such that

$$\frac{z(g*zf')'(z)}{(g*\Phi_1)(z)} \prec h(z).$$

Now there exists a $\Phi \varepsilon K_g(h)$ such that $z\Phi' = \Phi_1$ and hence the RHS of 3.4 is subordinate to $h(z)$ which implies the LHS of 3.4 is subordinate to $h(z)$ which in turn gives that $f \varepsilon Q_g(h)$.

REMARK 3.6: When $g(z) = z/1-z$ Theorem 3.8 reduces to Theorem 1 of Noor and Thomas [8]. By Theorems 3.2 (i) and 3.8 and from the observation we made just before Theorem 3.5 we get the following inclusions

$$\begin{tikzcd} K_g(h) \arrow[r] \arrow[d] \arrow[dr] & S_g(h) \\ Q_g(h) & C_g(h) \end{tikzcd}$$

where, the direction of the 'arrows' indicate the respective inclusions.

THEOREM 3.9: If $f \varepsilon Q_g(h)$, then for every convex univalent function $\Phi \varepsilon A$, $\Phi * f \varepsilon Q_g(h)$.

THEOREM 3.10: $Q_g(h) \subseteq Q_{\Phi * g}(h)$ for every convex univalent function $\Phi \varepsilon A$, in particular $Q_{a+1}(h) \subseteq Q_a(h)$ for $a > 1$.

PROOF: The proofs of the above Theorems 3.9 and 3.10 are omitted because it will follow from 3.8 and the corresponding theorems for the class $C_g(h)$.

4. CONCLUDING REMARKS.

It would be interesting to find a necessary and sufficient condition on the function $g(z)$ so that $f*g$ univalent implies f is univalent.

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