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A CHARACTERIZATION OF THE GRASSMANN MANIFOLD $G_{p,2}(\mathbb{R})$. ANOTHER REVIEW

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ABSTRACT. The purpose of the present study is to characterise the Grassmann manifold $G_{p,2}(R)$ and its non compact dual $G^*_{p,2}(R)$ by means of a particular parallel tensor field T of type (1,3) and the Weingarten map on geodesic spheres.

KEY WORDS AND PHRASES. Grassman manifold, Tensor field, Riemannian curvature tensor, Symmetric space, Eigenvector, Normal neighbourhood, Kahler manifold.
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1. PRELIMINARIES.

Let $G_{p,2}(R)$ be the oriented real Grassmann manifold of 2-planes in R^{p+2}. The purpose of this paper is to characterise this manifold and its non-compact dual $G^*_{p,2}(R)$ by means of a particular tensor field T of type (1,3) and the Weingarten map on geodesic spheres.

The problem was first considered by L. Vanhecke and T.J. Willmore who characterised spaces of constant curvature and spaces of constant holomorphic sectional curvature [1]. The case $G_{p,2}(R)$ has considered by the second of the authors in [2]. These results were generalised by the first author and D.E. Blair in [3], [4]. In this respect the conditions we need differ from those of [2] and, as our proof shows, some of the conditions given there are redundant.

We begin with some general remarks on Jacobi vector fields and geodesic spheres. Let M be a Riemannian manifold of dimension $n>2$ and let U be a normal neighbourhood of a point $m \in M$. We may take U to be a geodesic ball of radius r. Choose an orthonormal basis for the tangent space M_m and let $\{x^i\}$, $i=1,\dots,n$ be the corresponding normal coordinate system on U. Write N for the unit vector field on

$U-\{m\}$ tangent to geodesics from m, thus $N = \frac{x^i}{s}\frac{\partial}{\partial x^i}$ where s denotes geodesic

distance from m. Let V be the unit tangent field to a geodesic $\gamma:(-r,r)\to U$, with

$\gamma(0)=m$, choose a non-zero vector $W_m=a^i(\frac{\partial}{\partial x^i})_m$ normal to V_m and let $Y=a^i s\frac{\partial}{\partial x^i}$ on U.

Then on $\gamma-\{m\}$, we have $[Y,N]=0$ and $R(N,Y)N=\nabla_N\nabla_Y N=\nabla_N^2 Y$. Consequently the vector field

X on γ defined by $X_{\gamma(\sigma)}=a^i\sigma(\frac{\partial}{\partial x^i})_{\gamma(\sigma)}$, $-r<\sigma<r$, satifies

$$\nabla_X N=\nabla_N X \tag{1.1}$$

on $\gamma-\{m\}$ and, by continuity,

$$R(V,X)V=\nabla_V^2 X \text{ on } \gamma. \tag{1.2}$$

Thus X is a Jacobi vector field on γ for which

$$X_m=0 \text{ and } \nabla_{V_m} X=W_m \tag{1.3}$$

In particular, X is normal to V and, for any point Q on γ the normal space to V_Q is formed by evalutating all such Jacobi vector fields at Q. Now write $A=-\nabla_N$. For any geodesic sphere S in U with centre m, the restriction of A to tangent vecotrs to S is just the Weingarten map with respect to N as unit normal vector field. Also by (1.1) - (1.2) we have on $\gamma-\{m\}$

$$R(N,X)N=-\nabla_N AX=A^2X-(\nabla_N A)X \tag{1.4}$$

This equation is linear in X, hence, from the above remarks, it is valid for arbitrary vector fields X on $U-\{m\}$, where we note from the definition of A that $AN=0$.

Now suppose M is a Riemannian locally symmetric space. With the previous notation, suppose W_m satisfies

$$R(V_m,W_m)V_m=c.W_m$$

Let X be the Jacobi vector field on γ satisfying (1.3) and W the parallel vector field on γ with initial value W_m. Since $\nabla R=0$ we have $R(V,W)V=cW$ from which fW is a Jacobi vector field on γ with the same initial conditions (1.3) as X when we choose

$$f(\sigma) = \begin{cases} |c|^{-1/2}\sin(|c|^{1/2}\sigma), & \text{if } c < 0 \\ c^{-1/2}\sinh(c^{1/2}\sigma), & \text{if } c > 0 \\ \sigma & \text{, if } c = 0 \end{cases}$$

Thus X=fW and as a consequence of (1.1) and the definition of A

$$AW = -\frac{Nf}{f}W. \tag{1.5}$$

Since the Riemannian curvature at m is bounded, the set of eigenvalues c of $R(V_m,-)V_m$ taken over all unit vectors V_m is bounded, say $|c| < k^2, k > 0$. Thus

if we take U to be a geodesic ball of radius $< \frac{\pi}{k}$, then $f \neq 0$ on γ-{m}. We now have the following immediate consequence of (1.5).

PROPOSITION 1.1. Let m be a point in a Riemannian locally symmetric space of dimension > 2. Then m has a normal neighbourhood U such that, for each unit vector $V_m \in M_m$ and corresponding geodesic γ, the parallel translate of an eigenspace of the linear map $R(V_m,-)V_m$ along γ is contained in an eigenspace of the Weingarten map for each geodesic sphere in U with centre m.

2. STATEMENT OF MAIN THEOREM.

We consider the Grassmann manifold $G_{p,2}(R)$ as the homogeneous Riemannian symmetric space SO(p+2)/SO(p)xSO(2). The tangent space at any point $m \in G_{p,2}(R)$ can be identified with the vector space $M_{(px2)}$ of all px2 matrices over IR, considered as real vector space with inner product

$$g(X,Y) = \langle X,Y \rangle = trXY^t \tag{2.1}$$

which is clearly Hemitian with respect to the almost complex structure J given by $J(X_1,X_2) = (-X_2,X_1)$ where X_1,X_2 are column vectors of the form px1. An invariant Kaehler metric g is then defined on $G_{p,2}(R)$ and the corresponding Riemannian curvature tensor at m is represented by its action on $M_{(px2)}$ by ([3], p. 180)

$$R(X,Y)Z = XY^tZ-YX^tZ-Z(X^tY-Y^tX) \tag{2.2}$$

Similarly, for the non-compact dual $G^*_{p,2}$, the curvature tensor is just the negative of this, and it will be sufficient to consider the compact case. Of course the metric g can be replaced by any metric homothetic to it without affecting R.

The tensor T of type (1,3) defined at m by

$$T(X,Y,Z) = XY^tZ \tag{2.3}$$

is invariant by the isotropy group and so extends to a parallel vector field on $G_{p,2}(R)$, also denoted by T. We define linear endomorphisms T_{XY}, T^{XY} and T_X^Y at m by

$$T_{XY}Z=T(X,Y,Z),\ T^{XY}Z=T(Z,X,Y),\ T_X^YZ=T(X,Z,Y) \tag{2.4}$$

which are self-adjoint. Then one easily verifies that T has the following properties at m, hence on $G_{p,2}(R)$:

P_1 : $T(T(X,Y,Z),U,V) = T(X,T(U,Z,Y),V) = T(X,Y,T(Z,U,V))$

P_2 : $\langle T(X,Y,Z),W \rangle = \langle T(Z,W,X),Y \rangle = \langle T(Y,X,W),Z \rangle$

P_3 : For each unit vector X

i. $\mathrm{tr}\, T_{XX}=2$, ii. $\mathrm{tr}\, T_X^X = 1$, iii. $\mathrm{tr}\, T^{XX}=p$, $p\varepsilon Z^+$

Moreover it is known that $\dim G_{p,2}(R)=2p$. Particular use will be made of unit vectors X at m satisfying $T(X,X,X)=X$. Such vectors are characterised by the following:

LEMMA 2.1. Suppose $X=(X_1,X_2) \varepsilon M_{(px2)}$ satisfies $\mathrm{tr}\, XX^t=1$. Then $XX^tX=X$ if and only if X_1 and X_2 are linearly dependent.

PROOF. From the equation $XX^tX=X$ we easily get $||X_1||^2 . ||X_2||^2 (1-\cos^2\nu) = 0$, where ν is the angle between X_1, X_2, from which we have $\nu=0$. Conversely suppose $X_2=\lambda X_1$, $\lambda\varepsilon R$ therefore $(1+\lambda^2)||X_1||^2=1$ from which we have $XX^tX=X$.

Now choose a geodesic γ through m with unit tangent vector field V such that $T(V,V,V)=V$ on γ. This relation holds if and only if it is satisfied at m, and clearly such vectors exist at m. Then by (2.2) we have:

$$R(V,JV)V= -JV , \tag{2.5}$$

so using Proposition 1.1 we have the following result.

PROPOSITION 2.2. Let $m\varepsilon G_{p,2}(R)$ and choose a normal neighbourhood U of m as in Proposition 1.1. and let γcU be any geodesic ray from m with unit tangent vector N_m satisfying $T(N_m,N_m,N_m)=N_m$. Then the Weingaren map A has the following property

$$AJN_m=f(N_m).JN_m,\ f(N_m) \varepsilon R \tag{2.6}$$

We now state our main result.

THEOREM 2.3. Let M be a complete simply connected Kahler manifold of dimension $2p>2$ with metric g and almost complex structure J. Let T be a parallel tensor field of type (1,3) on M satisfying P_1 trough P_3. Suppose for each $m\in M$ there exists a normal neighbourhood U of m such that for each geodesic sphere S in U centred at m and for each unit normal N_m to S with $T(N_m,N_m,N_m)=N_m$ the Weingarten map satisfies (2.6). Then M is homothetic to either the Euclidean space E^{2p}, $G_{p,2}(R)$ or $G^*_{p,2}(R)$.

3. A CHARACTERISATION OF T ON $M_{(px2)}$

The proof of the Theorem depends largely on a characterisation of the structure described earlier on the tangent space to $G_{p,2}(R)$ at any point. For this purpose we require the following result.

PROPOSITION 3.1. Let V be a real finite dimensional vector space with inner product $\langle , \rangle$ and let T be a tensor of type (1,3) on V satisfying P_1 through P_3. Suppose $\dim V=2p > 2$, then there is a linear isomorphism of V onto $M_{(px2)}$ of all real px2 matrices considered as vector space and under identification $T(X,Y,Z)=XY^tZ$ and $\langle X,X\rangle=\mathrm{tr}XX^t$.

The proof of this proposition requires several lemmas. The first of these lemmas provides a useful duality between T_{XY} and T^{XY} and is immediate from P_1,P_2,P_3.

LEMMA 3.2. Define a tensor S on V by $S(X,Y,Z)=T(Z,Y,X)$ and write $S_{XY}=T^{YX}$, $S^{XY}=T_{YX}$, $S^X_Y=T^Y_X$. Then P_1,P_2 are satisfied when T is replaced by S and P_3 is satisfied when T^{XX} and T_{XX} are replaced by S_{XX} and S^{XX} respectively provided p and 2 are interchanged.

In what follows we remark that P_1 and P_2 may be used occasionally without reference.

LEMMA 3.3. For each non-zero $X\in V$ the linear endomorphisms T_{XX},T^{XX} and T^X_X are self-adjoint and $T(X,X,X)\neq 0$.

PROOF. The self-adjoint properties are clear from P_1. Also from P_3(i) there exist Y such that $T(X,X,Y)\neq 0$. Therefore from P_1 and P_2 we have:

$$0<\langle T(X,X,Y),\ T(X,X,Y)\rangle=\langle T(X,X,T(X,X,Y)),Y\rangle=\langle T(T(X,X,X),X,Y),Y\rangle$$

Thus $T(X,X,X)\neq 0$.

LEMMA 3.4. Suppose $X,Y\in V$ are non-zero and $T(X,X,Y)=\lambda.Y$. Then im T_{YY} is contained in the λ-eigenspace of T_{XX}. If $T(X,X,X)=\lambda X$ then λ is the only non-zero eigenvalue of T_{XX}.

PROOF. We have to prove that for any $Z\in V$, $T_{XX}(T(Y,Y,Z))=\lambda T(Y,Y,Z)$. In fact,

$$T_{XX}(T(Y,Y,Z))=T(X,X,T(Y,Y,Z))=T(T(X,X,Y),Y,Z)=\lambda T(Y,Y,Z).$$

Suppose now that there exist $\mu, Z\neq 0$ such that $T_{XX}Z=\mu Z$ then $T_{XX}(\frac{1}{\mu}Z) = Z$, so $Z\varepsilon \mathrm{im}T_{XX}$ and from the first part of the lemma $Z\varepsilon\lambda$-eigenspace of T_{XX}. Therefore $T_{XX}Z=\lambda Z$, so $\lambda Z=\mu Z$ and then $\mu=\lambda$.

From now on we use the following notation: Define $D \subset V$ by $X\varepsilon D$ if and only if $X=0$ or $\mathrm{rk}T_{XX}=\min\{\mathrm{rk}T_{YY}:Y\varepsilon V$ and $Y\neq 0\}$. Then for each non-zero $X\varepsilon D$ write $V_X=\mathrm{im}T_{XX}$.

Dually we define $D'\subset V$ by replacing T_{XX},T_{YY} above by T^{XX},T^{YY} and writing $V^X=\mathrm{im}\ T^{XX}$ for $X\neq 0$. Finally, we write $V_X^X= V_X\cap V^X$.

LEMMA 3.5. Let X and Y be non-zero vectors such that $X\varepsilon D$ and $Y\varepsilon V_X$. Then (i) $V_X\subset D$, (ii) $V_X=V_Y$, (iii) $T(X,X,X)=k||X||^2X$, where $k||X||^2\mathrm{rk}T_{XX}=2$ and $k=\max\{\nu/T(Z,Z,Z)= \nu Z,\ ||z||=1\}$, conversely, any vector U satisfying this equation belongs to D. (iv) $T_{XX}(V_X^{\perp}) =0$, where $V_X^{\perp}$ is the orthogonal complement of V_X in V.

PROOF. We may assume that $||X|| = ||Y|| = 1$. As a consequence of Lemmas 3.3 and 3.4, T_{XX} has exactly one nonzero eigenvalue, say λ possibly with multiplicity > 1. Since $T(X,X,Y)= \lambda Y$ then from the definiton of X and Lemma 3.4 $\mathrm{im}T_{YY} \subset \lambda$-eigenspace of T_{XX}, therefore $\mathrm{rk}T_{YY}=\dim \mathrm{im}\ T_{YY} \leq \dim(\lambda\text{-eigenspace of } T_{XX})=\mathrm{rk}T_{XX}$; but $\mathrm{rk}T_{XX}$ is a minimum, thus $\mathrm{rk}T_{YY}=\mathrm{rk}T_{XX}$, so $Y\varepsilon D$ and T_{YY} has a unique non-zero eigenvalue ν and $\mathrm{im}T_{XX}=\mathrm{im}T_{YY}$, which proves (i) and (ii). From the last equation we have $\mathrm{rk}T_{XX}=\mathrm{rk}T_{YY}=r$, suppose T_{XX} has the λ eigenvalue and T_{YY} the ν eigenvalue, then $\mathrm{tr}_{XX}=$ summation of eigenvalues $=r.\lambda=r.\nu$, so $\nu=\lambda$ and therefore $T(Y,Y,Y)= \nu .Y$. Next, let X_1 be the orthogonal projection of X onto the λ-eigenspace of T_{XX} the $T_{XX}X_1=\lambda X_1$. Let $X=X_1+X_2$ such that X_1 belongs to the λ-eigenspace and X_2 to the 0-eigenspace. Then $T_{XX}X=T_{XX}X_1+T_{XX}X_2=\lambda X_1$. Therefore $X_1\neq 0$ because if $X_1=0$ then $T_{XX}X=0$ which is impossible because we proved that $T_{XX}X \neq 0$. Furthermore,

$$\lambda^3||X_1||^2X_1= \lambda^2T(X_1,X_1,X_1) = T(\lambda X_1,\lambda X_1,X_1)=T(T(X,X,X),T(X,X,X),X_1) =$$

$$= T(X,X,T(X,T(X,X,X),X_1)) = T(X,X,T(X,X,T(X,X,X)))=$$

$$= \lambda^2T(X,X,X_1) = \lambda^3X_1$$

Thus $||X_1|| = 1$ so $X=X_1$ and $T(X,X,X) = \lambda X$. Since $\mathrm{rk}T_{XX}=2$ and is a minimum the first part of (iii) follows. Conversely, if $T(U,U,U) =k||U||^2U$ then $\mathrm{rk}T_{UU}=\mathrm{rk}T_{XX}$ so $U\varepsilon D$ as required. Finally (iv) is immediate since T_{XX} is self-adjoint and V_X is the k-eigenspace of T_{XX}.

LEMMA 3.6. If $Y\varepsilon V_X$ and $U,W\varepsilon V$ then $T(Y,U,W)\varepsilon V_X$.

PROOF. There exist $Z\varepsilon V$ such that $T(X,X,Z)=Y$. Hence from P_2, $T(Y,U,W)=T(T(X,X,Z),U,W)=T(X,X,T(Z,U,W))\varepsilon V_X$.

In the rest of this section let U be a unit vector in D. Then for any

$X,Y \in V_U^U$, $T(X,X,Y)=T(Y,X,X)=k||X||^2$. Y and linearisation gives, in paticular, $T(X,Y,X)+T(Y,X,X)=2k\langle X,Y\rangle X$. These equations imply.

LEMMA 3.7. For all $X,Y\varepsilon V_U^U, T_X^X Y=2k\langle X,Y\rangle X-k||X||^2 Y$. On the other hand we have the following result for $(V_U^U)^\perp$.

LEMMA 3.8. If $X\varepsilon V_U^U$ and $Y\varepsilon(V_U^U)^\perp$ then $T_X^X Y=0$.

PROOF. Since T_X^X is self adjoint it is sufficient to prove $(T_X^X)^2\ Y=0$. Let $Z\varepsilon V$, then $(T_X^X)^2 Z=T_X^X(T_X^X Z)=T_X^X(T(X,Z,X))= T(X,T(X,Z,X), X) =T(X,X,T(Z,X,X))=T_{XX}T^{XX}Z=T^{XX}T_{XX}Z$, so $(T_X^X)^2\ Z\varepsilon V_U^U$. Hence $\langle (T_X^X)^2 Y, Z \rangle = \langle Y, (T_X^X)^2 Z \rangle = 0$ which proves the Lemma.

LEMMA 3.9. (i) For any non-zero vector $X\varepsilon V_U^U$, $V_X^X=V_U^U=T_X^X(V_X^X)$ (ii) $k=1$, (iii) if $Y\varepsilon D$ is non zero then $\dim V_Y^Y=1$.

PROOF. From Lemma 3.5 (ii) and its dual $V_X^X=V_U^U$ and $V_U^U=T_X^X(V_X^X)$ from Lemma 3.7, this proves (i). From Lemmas 3.7 and 3.8 the non-zero eigenvalues of T_U^U are k and -k with mutiplicity 1 and d-1, where $d=\dim V_U^U$. Therefore $k-k(d-1)=1$ or $k(2-d)=1$ and due to the fact that $k > 0$ and d is an integer we conclude that $d=1$, therefore $k=1$ and $\dim V_U^U=1$. This proves (ii) and (iii) follows since the choice of unit vector $U\varepsilon D$ is arbitrary.

LEMMA 3.10. Suppose X,Y are unit vectors in V_U with Y orthogonal to V_X^X. Then

$$\text{(i) } \langle V_X^X, V_Y^Y \rangle = 0, \quad \text{(ii) } T(V_X^X, V_Y^Y, V) = 0$$

PROOF. Let $V\varepsilon V_X^X$ and $W\varepsilon V_Y^Y$. Then from Lemma 3.6 and its dual $\langle T(X,V,X),T(Y,W,Y) \rangle = \langle T(W,Y,T(X,V,X)),Y \rangle = \langle T(W,T(V,X,Y),X),Y \rangle = 0$ and (i) follows using Lemma 3.9 (i). Next for $Y\varepsilon V$, $\langle T(X,Y,V),T(X,Y,V) \rangle = \langle T(Y,X,T(X,Y,V)),V \rangle = \langle T(T(Y,X,X),Y,V),V \rangle$ Now $T(X,X,Y)=Y$ so from Lemma 3.8 $T(Y,X,X)=T(T(X,X,Y),X,X)= (T_X^X)^2\ Y=0$. Hence $T(X,Y,V)=0$ and (ii) follows using (i).

LEMMA 3.11. V_U^U admits a multiplication, with respect to which, it is isomprophic to R.

PROOF. Define a bilinear operation on V_U^U by $X.Y=T(X,U,Y)$. We show that V_U^U becomes a real associative division algebra and the lemma follows using Frobenius Theorem. Clearly U is a unit vector because $k=1$. Also multiplication is associative since $(X.Y).Z=T(T(X,U,Y),U,Z)=T(X,U,T(Y,U,Z))=X.(Y.Z)$. Moreover any non-zero X has an inverse $||X||^{-2}X$ and the proof is complete.

PROOF OF PROPOSITION 3.1. From Lemmas 3.9 and 3.10 together with their duals V_U(resp. V^U) is an orthogonal direct sum of subspaces of the form V_X^X, $X\varepsilon V_U$ (resp. $X\varepsilon V^U$) each of dimension $\omega=1$. Since $k=1$, we obtain using Lemma 3.5 and its dual $\dim V_U=\mathrm{rk}T_{UU}=2$ and $\dim V^U=\mathrm{rk}T^{UU}=p$, $p\varepsilon Z^+$. For convenience of notation, write $U= e= e_{11}$. From Lemma 3.11 we may consider V_U^U as a 1-dimensional vector space over R with vectors f, $f\varepsilon R$. Next we may choose sets of orthogonal unit vectors

$\{e_{11},e_{12}\}\subset V_e$ and $\{e_{11},\ldots,e_{p1}\}\subset V^e$ such that $V_e= V_1^1\oplus V_1^2$ and $V^e=V_1^1\oplus\ldots\oplus V_p^1$ where $V_1^\alpha=V_{e_{1\alpha}}^{e_{1\alpha}}, V_i^1=V_{e_{i1}}^{e_{i1}}$ for $\alpha=1,2$; $i=1,\ldots,p$ and the direct sums are orthogonal. Now define $e_{i\alpha}=T(e_{i1},e,e_{1\alpha})$ for $i=1,\ldots,p$; $\alpha=1,2$ noting consistency when $i=1$ or $\alpha=1$. Then $e_{i\alpha}\varepsilon V_{e_{i1}}\cap V^{e_{1\alpha}}\subset D$ and we write $V_{e_{i\alpha}}^{e_{i\alpha}}=V_i^\alpha$. From Lemma 3.9 (iii) each V_i^α has dimension $\omega=1$. Also we note from Lemma 3.1 (ii) and its dual form that for $\alpha\neq\beta$ and $i\neq j$

$$T(e_{1\alpha},e_{1\beta},V_1^1) = T(V_1^1,\ e_{i1},\ e_{j1}) = \{0\}$$

and it follows easily that V_i^α and V_j^β are orthogonal if $\alpha\neq\beta$ or $i\neq j$. Since $\dim V=2p$, V is the orthogonal direct sum of the subspaces V_i^α, $i=1,\ldots,p$; $\alpha=1,2$. Next for any $f\varepsilon R$ we define $e_{i\alpha}f=T(e_{i1},\ ef,\ e_{1\alpha})$. Then for $f,g,\ h\varepsilon R$,

$$T(e_{i\alpha}f,e_{j\beta}g,e_{k\gamma}h)=T(T(e_{i1},ef,e_{1\alpha}),e_{j\beta}g,e_{k\gamma}h)=T(e_{i1},T(e_{j\beta}g,e_{1\alpha},ef),$$

$$e_{k\gamma}h)=T(e_{i1},T(T(e_{j1},eg,e_{1\beta}),e_{1\alpha},ef),e_{k\gamma}h)=T(e_{i1},T(e_{j1},eg,$$

$$T(e_{1\beta},e_{1\alpha},ef)),e_{k\gamma}h)=\delta_{\alpha\beta}T(e_{i1},T(e_{j1},eg,ef),e_{k\gamma}h)=\delta_{\alpha\beta}T(e_{i1},ef,$$

$$T(eg,e_{j1},e_{k\gamma}h))=\delta_{\alpha\beta}T(e_{i1},ef,T(eg,e_{j1},T(e_{k1},eh,e_{1\gamma})))=$$

$$=\delta_{\alpha\beta}T(e_{i1},ef,T(T(eg,e_{j1},e_{k1}),eh,e_{1\gamma}))=\delta_{\alpha\beta}\delta_{jk}T(e_{i1},ef,T(eg,eh,e_{1\gamma}))=$$

$$=\delta_{\alpha\beta}\delta_{jk}T(e_{i1},T(eh,eg,ef),e_{1\gamma}).$$ Now,

$$ehgf=T(T(eh,e,eg),e,ef)=T(eh,T(e,eg,e),ef)=T(eh,eg,ef)$$

hence

$$T(e_{i\alpha}f,e_{j\beta}g,e_{k\gamma}h)=T(e_{i1},ehgf,e_{1\gamma})\delta_{\alpha\beta}\delta_{jk}=fgh\delta_{\alpha\beta}\delta_{jk}e_{i\gamma} \tag{3.1}$$

Since each $V_i^\alpha=T(e_{i1},V_1^1,e_{1\alpha})=e_{1\alpha}R$ it follows that V can be considered as a vector space over R with basis $\{e_{i\alpha}\}$, $i=1,\ldots,p;\alpha=1,2$. Then by considering $M_{(p\times2)}$ as a vector space over R we have an R-linear isomorphism:

$$\phi:\ V\to M_{(px2)};\ \sum_{i,\alpha}e_{i\alpha}x_{i\alpha}\to(x_{i\alpha})$$

From (3.1)

$$T(e_{i\alpha}x_{i\alpha}, e_{j\beta}y_{j\beta}, e_{k\gamma}z_{k\gamma}) = x_{i\alpha}y_{j\alpha}z_{j\gamma}\cdot e_{i\gamma}$$

Thus, if elements of V are represented by their corresponding matrices then T(X,Y,Z) coresponds to XY^tZ. Finally, using Lemma 3.11,

$$< e_{i\alpha}x_{i\alpha}, e_{j\beta}x_{j\beta} > = < T(e_{i1}, ex_{i\alpha}, e_{1\alpha}), e_{j\beta}x_{j\beta} > = < T(e_{1\alpha}, e_{j\beta}x_{j\beta}, e_{i1}),$$

$$, ex_{i\alpha} > = < ex_{i\alpha}, ex_{i\alpha} > = x_{i\alpha}\cdot x_{i\alpha} = \text{tr } XX^t,$$ and the proof is complete.

REMARK 3.12. Proposition 3.1 has a dual form obtained essentially by exchanging p,2 and replacing T by S as defined in Lemma 3.2. Thus write each basic vector $e_{i\alpha}$ as $\varepsilon_{\alpha i}$ and write any $X\varepsilon V$ as $\varepsilon_{\alpha i}x_{\alpha i}$. Then an R-linear isomorphism

$\psi: V \to M_{(2xp)}$ is defined by $\varepsilon_{\alpha i}x_{\alpha i} \to (x_{\alpha i})$; clearly $\psi = t\circ\phi$ where $t: M_{(2xp)} \to M_{(2xp)}$ is the transpose. If elements of V are represented by their corresponding matrices in $M_{(2xp)}$ then S(X,Y,Z)=T(Z,Y,X) corresponds to XY^tZ and $< \varepsilon_{\alpha i}x_{\alpha i}, \varepsilon_{\beta j}x_{\beta j} > = \text{tr } XX^t$.

4. PROOF OF THE MAIN THEOREM.

Before proving the Theorem we require some further lemmas. In what follows we denote $D=\{X\varepsilon V/T(X,X,X)=||X||^2 X\}$ and write as $e_{i\alpha}$ the matrix in $M_{(px2)}$ with 1 in row i column α and zeros elsewhere.

LEMMA 4.1. Let $X\varepsilon V$ be non-zero. Then (i) $X\varepsilon D$ if and only if $\phi(X)$ has rank one. (ii) $||X||=1$ and $X,Y\varepsilon D$ then $X+Y\varepsilon D$ if and only if $Y\varepsilon V_X \cup V^X$.

PROOF. (i) Elementary considerations show that if $A\varepsilon M_{(px2)}$ is non-zero then $AA^tA=(\text{tr } AA^t)\, A$, if and only if A has rank one. Analogously we conclude (ii).

LEMMA 4.2. Let R be a tensor of type (1,3) on V with the symmetry properties of a Riemanian curvature tensor and satisfying <R(JX,JY)Z,W >=<R(X,Y)Z,W> on V. Suppose for each $X\varepsilon D$ and $Y\varepsilon V$ orthogonal to X, <R(X,JX)X,JY>=0. Then the sectional curvature determined by R is constant on D.

PROOF. Write K(X) for the holomorphic sectional curvature for any unit vector $X\varepsilon V$. Also write R(X,Y,Z,W)=<R(Z,W)Y,X>. Now choose a unit vector $X\varepsilon D$. Let $Y\varepsilon V_X$ be a unit vector orthogonal to X. Then X+Y, $X-Y\varepsilon D$, so by hypothesis <R(X+Y,J(X+Y))) (X+Y), J(X-Y)> = 0 and it follows easy that K(X)=K(Y). If $X\varepsilon D$ then dim $V_x=2$ and there exist only one sectional curvature for {X,JX} and the case is trivial. If $X\varepsilon D$ then dim $V^X=p$. We prove that in this case we also have K(X)=K(Y) for all Y. Let Y be perpedicular to X and Y belongs to V^X. If p=2 then the case is obvious and we have K(X)=K(Y), if p > 2 then given X and any $Z\varepsilon V^X$ we have for any U perpedicular to X and Z,K(U)=K(Z) and K(U)=K(X), therefore K(X)=K(Z)=K(U). Thus K is constant on D, as required.

LEMMA 4.3. Define R as in Lemma 4.2 and suppose $R(X,JX)X=0$ and $R(X,Y)T=0$ for all $X, Y \in D$. Then $R=0$ on V.

PROOF. We first show for any $U \in D$, $R(V^U,V^U)V^U=0$. This depends only on the first of the above two conditions on R. Thus, by linearising the equation $R(X,JX)X=0$ we obtain, for all

$$X,Y \in V^U, R(X+Y,JX+JY)(X+Y)=0.$$

Therefore

$$R(X,JX)Y+2R(X,JY)X = 0 \tag{4.1}$$

Then (4.1) together with the Bianchi's identity applied to $R(X,JX)Y$, gives $R(X,Y)X = 0$. On replacing X, in this last equation, by X+Z it follows tht $R(X,Y)Z=0$ for all $X,Y, Z \in V^U$ as required. Clearly the same property holds with V_U replacing V^U. The second condition on R implies that, for any unit $U \in D$ and $X,Y \in V$

$$R(X,Y)U=R(X,Y)(T(U,U,U))=T(R(X,Y)U,U,U)+T(U,R(X,Y)U,U)+T(U,U,R(X,Y)U) \quad .$$

Then from Lemma 3.6 and its dual together with Lemma 3.8 we obtain $R(X,Y) \in V_U+V^U$. Next choose the basis $\{e_{i\alpha}\}$, $i=1,\dots,p$; $\alpha=1,2$ for V. We denote the subspace $V_{e_{i\alpha}}$ (resp. $V^{e_{i\alpha}}$) as V_i(resp. V^α). We must show that R acting on basis vectors is zero. Since the above properties of R still apply when U is replaced by any basis vector, we know that for $i=1,\dots,p$; $\alpha=1,2$ and $X,Y \in V$,

$$R(V_i,V_i)V_i=R(V^\alpha,V^\alpha)V^\alpha = 0, \text{ and} \tag{4.2}$$

$$R(X,Y)e_{i\alpha} \in V_i+V^\alpha \tag{4.3}$$

We now prove that each $R(V_i,V)V_i=0$. Clearly, $e_{j\alpha}+e_{j\beta} \in V_{e_{i\alpha}+e_{i\beta}}$ so by (4.2) and (4.3)

$$0=R(e_{i\alpha}+e_{i\beta},e_{j\alpha}+e_{j\beta})(e_{i\alpha}+e_{i\beta})=R(e_{i\alpha},e_{j\alpha})e_{i\beta}+R(e_{i\beta},e_{j\beta})e_{i\alpha}$$

But (4.3) implies that $R(e_{i\alpha},e_{j\alpha})e_{i\beta} \in V^\beta$ and $R(e_{i\beta},e_{j\beta})e_{i\alpha} \in V^\alpha$. It follows that

$$R(e_{i\alpha},e_{j\alpha})e_{i\beta}=0, \quad i,j=1,\dots p; \; \alpha,\beta=1,2 \tag{4.4}$$

Also if $i \neq j$ and $\alpha \neq \gamma$ then (4.3) implies that for all $X,Y \in V$

$$\langle R(e_{i\alpha},e_{j\gamma})X,Y \rangle = \langle R(Y,X)e_{j\gamma},e_{i\alpha} \rangle =0$$

Thus for $i \neq j$ and $\alpha \neq \gamma$

$$R(e_{i\alpha},e_{j\gamma}) = 0 \tag{4.5}$$

Then as a consequence of (4.4) and (4.5) each $R(V_i,V)V_i=0$. Since equations (4.2) and (4.3) are symmetric in V_i and V^α the same proof applies to give $R(V^\alpha,V)V^\alpha=0$ for $\alpha=1,2$. The Bianchi identity then shows that

$R(V_i,V_i)V=R(V^\alpha,V^\alpha)V=0$, and these two equations together with (4.5) prove that $R=0$ on V as required.

PROOF OF THEOREM 2.3. Under the conditions of the Theorem, suppose the unit vector $V_m \varepsilon M_m$ satisfies $T(V_m,V_m,V_m)=V_m$ and let V be the unit tangent vector field to the geodesic γ from m with initial tangent vector V_m. Then $T(V,V,V) = V$ along γ and from equation (2.5) $AJV=f(V)JV$ for some smooth function f on $\gamma-\{m\}$. It follows using equation (1.4) that if Y is a parallel vector field along γ normal to V then $\langle R(V,JV)V,JY\rangle=0$ on $\gamma-\{m\}$ and hence at m by continuity. Now consider M_m as the vector space V in Proposition 3.1. The tensor T at m satisfies P_1,P_2,P_3 and, with the notation of Lemma 4.2 for each $X\varepsilon D$ and Y orthogonal to X, $\langle R(X,JX)X, JY\rangle =0$. Hence from Lemma 4.2, K is a constant, say c on D, and for all unit vectors $X\varepsilon D$, $R(X,JX)X = -cJX$. Next, it is clear from Proposition 3.1 and equation (2.2) that a second curvature R_1 is defined on M_m by

$$R_1(X,Y)Z = T(X,Y,Z) + T(Z,Y,X)-T(Y,X,Z)-T(Z,X,Y) \tag{4.6}$$

and R_1 also satisfies the conditions of Lemma 4.2 with respect to the given almost complex structure J on M restricted to M_m. Moreover $R_1(X,JX)X = -JX$ for any unit vector $X\varepsilon D$. The tensor $R_2=R-cR_1$ then satisfies the conditions of Lemma 4.2 and Lemma 4.3 note that $R(X,Y)T=0$ since T is a parallel tensor field on M and $R_1(X,Y)T=0$ is the corresponding algebraic property of any point of $G_{p,2}(R)$. Thus by Lemma 4.3.

$$R=cR_1 \tag{4.7}$$

on M_m. But m is arbitrary so, defining R_1 on M by (4.6) we see that on M

$$R=fR_1 \tag{4.8}$$

for some function f, the Ricci tensor corresponding to R_1 is a multiple of the metric g, as can be seen either by direct computation ([6]) or by noting that $G_{p,2}(\mathbb{R})$ is an Eistein space. Hence from (4.8), (M,g) is an Eistein space and $f=c$ on M. Then $\nabla R_1=0$ implies $\nabla R=0$ so (M,g) is locally symmetric space.

Suppose $c=0$, then (M,g) is flat. Conversely on any flat Kahler manifold M we can define T by

$$T(X,Y,Z)=g(X,Y)Z+g(X,JY)Z. \tag{4.9}$$

With M complete and simply connected as in the theorem, M is isometric to Euclidean space E^{2p}. Next, suppose $c > 0$ and define g' and T' on (M,g) by $g'=cg$ and $T' = c.T$. Then the conditions of the theorem are satisfied with g',T' replacing g,T; further, since the curvature tensor R is unchanged by the homothety, we have from (4.6) and (4.7)

$$R(X,Y)Z=T'(X,Y,Z)+T'(Z,Y,X)-T'(Y,X,Z)-T'(Z,X,Y) \tag{4.10}$$

on (M,g'). We know that (M,g') is a locally symmetric space and it is clear from Proposition 3.1 and equations (2.1), (2.2) and (4.10) that the tangent spaces at any two points of $G_{p,2}(R)$ and M are related by a linear isomorphism which preserves inner products and the curvature tensors. Hence $G_{p,2}(R)$ and M are locally isometric ([5], p. 265) and this extends to a global isometry when M is complete and simply connected since $G_{p,2}(R)$ has these properties. Finally if $c < 0$ we clearly obtain the same result for the non compact dual of $G_{p,2}(R)$ and the proof is complete.

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