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ON THE NORMALIZER OF A GROUP IN THE CAYLEY REPRESENTATION

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ABSTRACT If G is embedded as a proper subgroup of X in the Cayley representation of G, then the problem of "if $N_X(G)$ is always larger than G" is studied in this paper.

KEY WORDS AND PHRASES: Cayley representation, Wreath product, permutation group.

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Let R be the Cayley representation (i.e., the right regular representation) of a group G given by $R(g) = \binom{x}{xg}$ for all $g \in G$ and $x \in G$. Under the mapping R, the group G is embedded into a subgroup $R(G)$ of the symmetric group S_Ω, the group of permutations on the set Ω consisting of the elements of the group G. We identify $R(G)$ with G and say that G is a subgroup of S_Ω. The centralizer of G in S_Ω consists precisely of the elements of the form $\binom{x}{gx}$. (See Lemma 1.) In particular, if G is abelian then G is self centralizing in S_Ω. Also, the normalizer of G in S_Ω is equal to $G \cdot Aut(G)$ where $Aut(G)$ is the full automorphism group of G (see Lemma 2).

Suppose that the group G is nonabelian. If X is a subgroup of S_Ω, containing a permutation of the type $\binom{x}{gx}$ for sme $g \in G - Z(G)$ such that the property

$$G \lneq X \leq S_\Omega \tag{*}$$

holds, then it follows that $N_X(G)$ contains G properly. However, it is easy to see that any element of S_Ω which normalizes G is not always a permutation of the form $\binom{x}{gx}$.

When the group G is abelian, the permutations $\binom{x}{gx}$ all lie in G and so G is self centralizing in S_Ω. In this way one cannot find a group X satisfying (*) by the above method. However, P. Bhattacharya [1] proved that if G is any finite, abelian p group satisfying (*) then $N_X(G) \gneq G$. P. Bhattacharya and N. Mukherjjee [2] also prove that if G is any finite, nilpotent, Hall subgroup of X satisfying (*) and the Sylow p subgroups of G are regular for all primes p dividing the order of G, then $N_X(G) \gneq G$. In other words, that X must contain an element of the outer automorphism group of G.

In this paper we will prove that if G is any abelian Hall subgroup of X, satisfying the condition (*) then $G \lneqq N_X(G)$. We will also give an example to show that the condition of being Hall subgroup is necessary in the above theorem. We will also show that if G is any nilpotent, Hall subgroup of X satisfying the condition (*) and the Sylow p subgroups P of G do not have a factor group that is isomorphic to the Wreath product of $Z_p \wr Z_p$ then $G \lneqq N_X(G)$. In particular it follows that if G is any finite p-group and does not have a factor group isormophic to $Z_p \wr Z_p$ then $G \lneqq N_X(G)$ [i.e., the condition being a Hall subgroup is not necessary]. As a corollary it also follows that if G is any regular p-group satisfying the conition (*) then $G \lneqq N_X(G)$. We will give an example to show that the condition of G having no factor group isomorphic to $Z_p \wr Z_p$ is necessary.

Lemma 1. Let R be the right regular representation of a finite group G and L, the left regular representation of G. Under the mappings L and R, the groups $L(G)$ and $R(G)$ are subgroups of S_Ω and $C_{S_\Omega}(R(G)) = L(G)$.

Proof: Let $\binom{x}{xg} \in R(G)$, $\binom{x}{hx} \in L(G)$

$$\begin{pmatrix} x \\ xg \end{pmatrix}\begin{pmatrix} x \\ hx \end{pmatrix} = \begin{pmatrix} x \\ xg \end{pmatrix}\begin{pmatrix} xg \\ hxg \end{pmatrix} = \begin{pmatrix} x \\ hxg \end{pmatrix}$$
$$= \begin{pmatrix} x \\ hx \end{pmatrix}\begin{pmatrix} hx \\ hxg \end{pmatrix} = \begin{pmatrix} x \\ hx \end{pmatrix}\begin{pmatrix} x \\ xg \end{pmatrix}.$$

Hence $L(G) \subseteq C_{S_\Omega}(R(G))$.

Now suppose $\binom{x}{x'} \in S_\Omega$ and $\binom{x}{x'}$ centralizes $R(G)$. So $\binom{x}{xg}\binom{x}{x'} = \binom{x}{xg}\binom{xg}{(xg)'} = \binom{x}{(xg)'}$ and $\binom{x}{x'}\binom{x}{xg} = \binom{x}{x'}\binom{x'}{x'g} = \binom{x}{x'g}$.

Since $\binom{x}{x'} \in C_{S_\Omega}$, so $x'g = (xg)'$ for all $x, g \in G$.

Hence $x' = (xg)'g^{-1}$. Now plug in $g = x^{-1}$. So $x' = 1' \cdot x$. Thus $\binom{x}{x'} = \binom{x}{1' \cdot x} \in L(G)$. Hence $C_{S_\Omega}(R(G)) = L(G)$.

Lemma 2: With the same notation as in Lemma 1, we have $N_{S_\Omega}(R(G)) = R(G) \cdot Aut(G)$.

Proof: Let $\binom{x}{x'} \in Aut(G)$ then $\binom{x}{x'} \in S_\Omega$,

$$\begin{pmatrix} x \\ x' \end{pmatrix}^{-1}\begin{pmatrix} x \\ xg \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} x' \\ x \end{pmatrix}\begin{pmatrix} x \\ xg \end{pmatrix}\begin{pmatrix} xg \\ (xg)' \end{pmatrix}$$
$$= \begin{pmatrix} x' \\ (xg)' \end{pmatrix} = \begin{pmatrix} x' \\ x'g' \end{pmatrix} = \begin{pmatrix} x \\ xg' \end{pmatrix} \in R(G).$$

Hence $Aut(G) \subseteq N_{S_\Omega}(R(G))$. Conversely, let $\binom{x}{x'}$ be an arbitrary element of $N_{S_\Omega}(G)$. Let $a = 1'$. So $\binom{x}{xa^{-1}} \in R(G)$. Let $O = \binom{x}{x'}\binom{x}{xa^{-1}}$. So O sends 1 to 1. Now $\binom{x}{x^O}^{-1}\binom{x}{xg}\binom{x}{x^O} \in R(G)$. So $\binom{x^O}{x}\binom{x}{xg}\binom{xg}{(xg)^O} = \binom{x^O}{(xg)^O} = \binom{x^O}{x^O g^*}$ since it lies in $R(G)$, i.e., $(xg)^O = x^O \cdot g^*$. Plug in $x = 1$, we get $g^* = g^O \Rightarrow (xg)^O = x^O \cdot g^O \Rightarrow O$ is an autormophism of $G \Rightarrow N_{S_\Omega}(R(G)) = R(G) \cdot Aut(G)$.

Lemma 3: Let G be any finite group satisfying the condition (*). Then for any $\alpha \in \Omega$

(i) $G \cap X_\alpha = \{e\}$.

(ii) $X = G \cdot X_\alpha$

(iii) X_α is core free, i.e., it does not contain any non-identify normal subgroup of X.

Proof: Recall that here G is identified with $R(G)$ in $G \leq X \leq S_\Omega$. Since R is the right regular representation of G, so $R(g)$ does not fix any $\alpha \in \Omega$ except when $g = e$. So $G \cap X_\alpha =$

$\{e\}$. Also X acts transitively on Ω, $|\alpha^X| = |\Omega| = |G|$. Now $[X : X_\alpha] = |\alpha^X| = |G|$. So $X = G \cdot X_\alpha$. For part (iii) suppose $N \triangleleft X$ and $N \subseteq X_\alpha$. So $N \subseteq \cap_{x \in X} x^{-1} X_\alpha x$, i.e., if n is an arbitrary element of N, then n can be written as $n = x^{-1}ux$ for all $x \in X$ and some $u \in X_\alpha$. Here u depends on x, i.e., $x \cdot n = u \cdot x$ or $\alpha^{xn} = \alpha^{ux} = \alpha^x$ since u fixes α, i.e., n fixes α^x for all $x \in X$, but X acts transitively on $\Omega \Rightarrow n$ fixes every element of $X \Rightarrow n = e \Rightarrow N = \{e\}$.

Lemma 4: (Core Theorem): Let H be any subgroup of G with $[G : H] = n$. then $G/core\,H$ is isomorphic to a subgroup of S_n where core H is the largest normal subgroup of G which is contained in H.

Proof: Let Ω be the set of distinct right cosets of H in G, i.e., $\Omega = \{Hg_1, Hg_2, \ldots, Hg_n\}$. Then the mapping σ defined by $\sigma(g) = \begin{pmatrix} Hg_i \\ Hg_ig \end{pmatrix}$ is a transitive permutation representation of G of degree n with Kernel of $\sigma = core\,H$.

Theorem 5: Let G be a finite abelian, Hall subgroup of X, satisfying the condition (*). Then $N_X(G) \gneq G$.

Proof: Suppose the result is false, i.e., there exists a subgroup X of S_Ω satisfying $G \lneq X \leq S_\Omega$ and $N_X(G) = G$. Amongst all subgroups of S_Ω containing G property, pick X to be smallest. In other words, G is a maximal subgroup of X. Let $|G| = p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots, p_t^{\alpha_t}$ with p_i distinct primes. Let P_i be Sylow p_i subgroups of G for $i = 1, 2, \ldots, t$. Since G is a maximal subgroup of X, so $N_X(P_i) = G$ or $N_X(P_i) = X$. Renumber the p_i's if necessasry and say $N_X(P_i) = G$ for $i = 1, \ldots, \ell$ and $N_X(P_i) = X$ for $i = \ell + 1, \ldots, t$. For $i = 1, \ldots, \ell$, $N_X(P_i) = C_X(P_i) = G$. So by Burnside Lemma X has a normal p_i complement. For $j = \ell + 1, \ldots, t$, $P_j \triangleleft X \Rightarrow C_X(P_j) \triangleleft X$ $G \subseteq C_X(P_j) \Rightarrow C_X(P_j) = X = N_X(P_j)$. So X has a normal P_i complement M_i for all $i \Rightarrow X_\alpha = \bigcap_{i=1}^{t} M_i$, $X_\alpha \triangleleft S$ which is a contradiction to Lemma 3.

In the case where G is abelian, but not Hall subgroup of X, the result is not true as illustrated by the following example.

Example 6: Let $X = Z_3 \times S_3 = \langle a \rangle \times \langle b, c | b^3 = c^2 = 1,\ c^{-1}bc = b^{-1} \rangle$.

Let $G = Z_3 \times Z_2 = \langle a \rangle \times \langle c \rangle \simeq Z_6$. Let H be the subgroup of X of order 3 generated by the ordered pair (a, b). Then H is not normal in X since (e, c) does not normalize H. So H is core free, of index 6 in X. By Lemma 4, $G \lneq X \leq S_6$. Now G is abelian, not Hall subgroup of X and $N_X(G) = G$.

Theorem 7: Let G be a finite, nilpotent, Hall subgroup of X, satisfying the condition (*). Suppose that the Sylow p subgroups P of G do not have a factor group isomorphic to the Wreath product of $Z_p \wr Z_p$ for all primes p dividing the order of G. Then $N_X(G) \gneq G$.

Proof: Suppose the result is false, i.e., there exists a subgroup X of S_Ω satisfying $G \lneq X \leq S_\Omega$ and $N_X(G) = G$. Amongst all subgroups of S_Ω containing G properly, pick X to be smallest. In other words G is a maximal subgroup of X. Let $|G| = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_t^{\alpha_t}$, here p_i are all distinct primes. Since G is nilpotent, so $G = P_1 \times P_2 \times \ldots \times P_t$ where P_i are Sylow p_i subgroups of G. So we have either $N_X(P_i) = G$ or $N_X(P_i) = X$. Renumber the p_i's if necessary and say $N_X(P_i) = G$ for $i = 1, \ldots \ell$ and $N_X(P_i) = X$ for $i = \ell + 1, \ldots, t$.

Let us look at the case $i = 1, \ldots, \ell$. We have $N = N_X(P_i) = G$. By Yoshida's transfer theorme [3], X has normal p_i complement M_i.

Let $M = \cap_{i=1}^{\ell} M_i$. So $p_i \nmid |M|$ for $i = 1, \ldots, \ell$. Now for $j = \ell + 1, \ldots, t$, $N_X(P_j) = X$. So $P_j \triangleleft X$ which implies that $C_X(P_j) \triangleleft X$ and $P_jC_X(P_j) \triangleleft X$ and $G \subseteq P_j \cdot C_X(P_j) \Rightarrow P_jC_X(P_j) = X$.

For $\alpha \in \Omega$, by Lemma 3 $X = G \cdot X_\alpha$; $G \cap X_\alpha = 1$; $(|G|, |X_\alpha|) = 1 \Rightarrow X_\alpha \subseteq C_X(P_j) \cap M \Rightarrow X_\alpha \subseteq C_M(P_j)$ for $j = \ell + 1, \ldots, t$. $|M| = p_{j+1}^{\alpha_{j+1}} \ldots p_t^{\alpha_t} \cdot |X_\alpha| \Rightarrow X_\alpha \Delta M \Rightarrow X_\alpha$ is a characteristic subgroup of $M \Delta G \Rightarrow X_\alpha \Delta G$, which is a contradiction to Lemma 3.

As an immediate corollary to the theorem, we get the result of P. Bhattacharya and N. Mukherjee [2].

Corollary 8: Let G be a finite, regular p subgroup of X and satisfies the condition (*), then $N_X(G) \gneqq G$.

Proof: If G is not a Hall subgroup of X then G is propertly contained in a Sylow p subgroup of X and so $N_X(G) \gneqq G$. So we can assume that G is a Hall subgroup of X. Now G being a regular p group $\Rightarrow G$ does not have a factor group isomorphic to $Z_p \wr Z_p$. So Theorem 7 proves the result.

Corollary 9. Let G be a finite, nilpotent, Hall subgroup of X, satisfying the condition (*). Suppose further that Sylow p subgroups of G are regular for all primes p dividing the order of G then $N_X(G) \gneqq G$.

Corollary 10: Let G be a finite p group, satisfying the condition (*). Suppose that G does not have a factor group isomorphic to $Z_p \wr Z_p$, then $G \lneqq N_X(G)$.

The condition that the Sylow p subgroups of G in Theorem 6 have the property that it has no homomorphic isomorphic to $Z_p \wr Z_p$ is necessary. Se example below.

Example: Let X be the simple group of order 168. Let $G \in Syl_2(X)$. Then $G \cong Z_2 \wr Z_2$ so G is nilpotent, Hall subgroup of X. Since H = the normalizer of a Sylow 7 subgroup has index 8, so by Lemma 4, $G \subseteq X \subseteq S_8$, i.e., G satisfies the condition (*) but $N_X(G) = G$.

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