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THE UNIFORM BOUNDEDNESS PRINCIPLE FOR ORDER BOUNDED OPERATORS

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ABSTRACT: Under appropriate hypotheses on the spaces, it is shown that a sequence of order bounded linear operators which is pointwise order bounded is uniformly order bounded on order bounded subsets. This result is used to establish a Banach-Steinhaus Theorem for order bounded operators.

KEY WORDS AND PHRASES: lattice, order bounded operator, uniform boundedness principle, Banach-Steinhaus Theorem.

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1. INTRODUCTION

In this note we consider the problem of obtaining a version of the classical Uniform Boundedness Principle of functional analysis for linear operators between vector lattices. If X is a Banach space and Y is a normed linear space, the Uniform Boundedness Principle then asserts that any sequence $\{T_i\}$ of continuous linear operators from X into Y which is pointwise bounded on X is such that the sequence of operator norms $\{\|T_i\|\}$ is bounded ([1] §4). It is easy to see that the condition that the $\{\|T_i\|\}$ are bounded is equivalent to the condition that the sequence $\{T_i\}$ is uniformly bounded on bounded subsets of X (see the discussion in [1] §4). Thus, a possible version of the Uniform Boundedness Principle for order bounded linear operators $\{T_i\}$ between vector lattices X and Y might be that if $\{T_i\}$ is pointwise order bounded on X, then $\{T_i\}$ is uniformly order bounded on order bounded subsets of X. We will show below in Example 1 that such a straightforward analogue of the Uniform Boundedness Principle does not hold even if both X and Y are Banach lattices. However, by imposing additional conditions on the spaces and by employing the matrix methods of [1], we will obtain an order version of the Uniform Boundedness Principle in Theorem 3 below. Using the Uniform Boundedness Principle, we also establish a version of the Banach-Steinhaus Theorem for order bounded operators which generalizes a result of Nakano.

2. RESULTS

First, consider the following example which shows that a straightforward analogue of the Uniform Boundedness Principle does not hold for order bounded linear operators between Banach lattices. (A linear operator T between vector lattices or Riesz spaces X and Y is order bounded if T carries order bounded subsets of X into order bounded subsets of Y, where a subset A of a vector lattice X is order bounded if there is an order interval $[-u, u] = \{v \in X : -u \le v \le u\}$ such that $A \subseteq [-u, u]$ (in general, we conform to the notation and terminology of [5]).)

EXAMPLE 1. Let $X = L^1[0,1]$ and $Y = c_0$ and assume that these spaces have the usual pointwise ordering. For $f \in L^1[0,1]$, set $f_k = \int_0^1 f(t)\sin \pi kt\,dt$. Define $T_k : X \to Y$ by $T_k f = \sum_{j=1}^{k} f_j e_j$, where e_j is the element of c_0 with a 1 in the j^{th} coordinate and 0 in the other coordinates. Each T_k is order bounded since if $|f| \le g$ in X, then $|f_j| \le \int_0^1 g = a$ and $|T_k f| \le \sum_{j=1}^{k} a e_j$. Also, the sequence $\{T_k\}$ is pointwise order bounded on X since if $f \in X$, then $|T_k f| \le \sum_{j=1}^{\infty} |f_j| e_j \in c_0$ (note $\{|f_j|\} \in c_0$ by the Riemann-Lebesgue Lemma). However, $\{T_k\}$ is not uniformly order bounded on order bounded subsets of X since if $\varphi_k(t) = \sin \pi kt$, then $\{\varphi_k\}$ is order bounded in X ($|\varphi_k| \le 1$) but $\{T_i(\varphi_i)\} = \{e_i/2\}$ is not order bounded in c_0.

Note that both X and Y in this example are Banach lattices and both are Dedekind complete.

In order to obtain our version of the Uniform Boundedness Principle for ordered spaces, we first obtain a matrix theorem which is the analogue for ordered spaces of the matrix result given in [1] 2.1.

Throughout the remainder of this note we let X and Y denote Riesz spaces (vector lattices). A sequence $\{x_k\}$ in X is <u>u-convergent</u> to x, where $u \ge 0$, if there exists a scalar sequence $t_k \downarrow 0$ such that $|x_k - x| \le t_k u$; we write $u - \lim x_k = x$. The element u is called a <u>convergence regulator</u> for $\{x_k\}$. The sequence $\{x_k\}$ is <u>relatively uniformly convergent</u> to x if $\{x_k\}$ is u-convergent to x for some convergence regulator $u \ge 0$; we write $r - \lim x_k = x$.

LEMMA 2. ([4] 2.1) Let the infinite matrix $[x_{ij}]$, $x_{ij} \in X$, be such that its rows and columns are u-convergent to 0. Let $\epsilon_{ij} > 0$, $\epsilon_{ij} \in \mathbb{R}$. Then there exists an increasing sequence of positive integers $\{p_i\}$ such that $|x_{p_ip_j}| \le \epsilon_{ij}u$ for $i \ne j$.

PROOF. Put $p_1 = 1$. Since $\{x_{1j}\}$ and $\{x_{i1}\}$ are u-convergent to 0, there exists $p_2 > p_1$ such that $|x_{1p_2}| \le \epsilon_{12}u$ and $|x_{p_21}| \le \epsilon_{21}u$. Similarly, there is $p_3 > p_2$ such that $|x_{p_1p_3}| \le \epsilon_{13}u$, $|x_{p_2p_3}| \le \epsilon_{23}u$, $|x_{p_3p_2}| \le \epsilon_{32}u$ and $|x_{p_3p_1}| \le \epsilon_{31}u$. Now continue.

We now prove our Uniform Boundedness Principle for order bounded operators. Recall that if Y is Dedekind complete, then a linear map $T : X \to Y$ is order bounded if and only if T is regular ([5] VIII.2.2).

THEOREM 3. Let X be Dedekind σ-complete and let Y be Dedekind complete and have an order unit u. If $T_i : X \to Y$ is a sequence of order bounded linear maps which is pointwise order bounded on X, then $\{T_i\}$ is uniformly order bounded on order bounded subsets of X.

PROOF. If the conclusion fails, there is an interval $[-w, w]$ in X such that $\{T_i([-w, w]) : i \in \mathbb{N}\}$ is not order bounded. Thus, for each i there exist $x_i \in [-w, w]$ and m_i such that $T_{m_i}x_i \notin i^4[-u, u]$. For notational convenience, assume that $m_i = i$. Then

$$T_ix_i \notin i^4[-u, u]. \tag{2.1}$$

Now consider the matrix $M = [(1/i)T_i(x_j/j^2)]$. For each j, $\{T_i(x_j/j^2)\}_i$ is order bounded so the j^{th} column of M is relatively uniformly convergent to 0. The sequence $\{x_j/j^2\}$ is relatively uniformly convergent to 0, and since each T_i is sequentially continuous with respect to relative uniform convergence ([5] VIII 1.2), the i^{th} row of M is also relatively uniformly convergent to 0. Thus, the rows and columns of M are u-convergent to 0. By Lemma 2 there is an increasing sequence of positive integers $\{p_i\}$ such that $|(1/p_i)T_{p_i}(x_{p_j}/p_j{}^2)| \le 2^{-i-j}v$ for $i \ne j$. Again for notational convenience, we assume that $p_i = i$. Since X is σ-complete and $|x_j| \le w$, the series $\sum x_j/j^2$ is absolutely order convergent to an element $x \in X$ ([5] IV. 9); moreover, this series is actually w-convergent in X since $|x - \sum_{j=1}^{n} x_j/j^2| \le \sum_{j=n+1}^{\infty} |x_j|/j^2 \le (\sum_{j=n+1}^{\infty} 1/j^2)w$. From the continuity of T_i with respect to relative uniform convergence, we have

$$|(1/i)T_i(x_i/i^2)| \le |\sum_{j=1}^{\infty}(1/i)T_i(x_j/j^2)| + |\sum_{\substack{j=1\\j\ne i}}^{\infty}(1/i)T_i(x_j/j^2)|$$

$$\le |(1/i)T_i(x)| + \sum_{\substack{j=1\\j\ne i}}^{\infty}|(1/i)T_i(x_j/j^2)|$$

$$\le |(1/i)T_i(x)| + \sum_{\substack{j=1\\j\ne i}}^{\infty}2^{-i-j}v$$

$$\le (1/i)|T_i(x)| + 2^{-i}v \ , \qquad (2.2)$$

where we have used the order completeness of Y to insure that the series on the right hand side of (2) are convergent. Both terms on the right hand side of (2) are order bounded. Hence, there exists k such that $(1/i)|T_i(x_i/i^2)| \in k[-u, u]$ for all i. Putting $i = k$ contradicts (1) and the result is established.

Note that the range space c_0 in Example 1 is Dedekind complete, but does not have an order unit.

It is perhaps worthwhile noting that if $\{T_i\}$ is uniformly order bounded on order bounded subsets of X, then the sequence of moduli $\{|T_i|\}$ also has this property. For if $0 \le x \le u$ and $|T_i[0, u]| \le w$, then $|T_i|x = \sup\{|T_iz| : 0 \le z \le x\} \le w$ ([5] VIII.2) so $|T_i|[0, u] \subseteq [0, w]$.

From Theorem 3 we can also obtain an order analogue of the equicontinuity conclusion of the classical Uniform Boundedness Principle. Recall that if $\{T_i\}$ is a sequence of continuous linear operators from a normed space X into a normed space Y, then $\{T_i\}$ is equicontinuous if and only if $\{T_i\}$ is uniformly bounded on bounded subsets of X if and only if $T_ix_i \to 0$ whenever $x_i \to 0$ if and only if $\lim_j T_ix_j = 0$ uniformly in i whenever $x_j \to 0$ ([1] 4.5). It is easy to establish order analogues of these equicontinuity conditions for operators which satisfy the conclusion of Theorem 3. We give order analogues of these conditions below and consider the relationship between them. If for each $a \in A$, $\{x_{ia}\}$ is a sequence in X, we say that <u>$\{x_{ia}\}$ is u-convergent to 0 uniformly for $a \in A$</u> if there exists a sequence $t_i \downarrow 0$ such that $|x_{ia}| \le t_iu$ for all $a \in A$; we write $u - \lim_i x_{ia} = 0$ uniformly for $a \in A$.

PROPOSITION 4. Let $T_i : X \to Y$ be order bounded. Consider

(i) $\{T_i\}$ is uniformly order bounded on order bounded subsets of X,

(ii) if $r - \lim x_j = 0$, then $v - \lim_j T_i x_j = 0$ uniformly for $i \in \mathbb{N}$ for some $v \in Y$, $v \geq 0$,

(iii) $r - \lim T_i x_i = 0$ whenever $r - \lim x_i = 0$.

Always (i) implies (ii) implies (iii). If Y is Archimedean and has the boundedness property ([3] 1.5.12), then (iii) implies (i).

PROOF. Assume (i) holds. To establish (ii), there exists $t_j \uparrow \infty$ such that $\{t_j x_j\}$ is relatively uniformly convergent to 0 ([4] VI.4). Since $\{t_j x_j\}$ is order bounded, by (i) there is $v \in Y$ such that $|T_i(t_j x_j)| \leq v$ for all i, j. Then $|T_i x_j| \leq (1/t_j)v$ implies (ii).

Clearly (ii) implies (iii). Assume (iii) and that Y is Archimedean with the boundedness property. Let $u \geq 0$, $u \in X$. By the boundedness property, it suffices to show that if $\{T_{k_i}\}$ is a subsequence, $|x_i| \leq u$ and $t_i \downarrow 0$, then $r - \lim t_i T_{k_i} x_i = 0$. But, since $u - \lim t_i x_i = 0$, this follows from (iii).

Theorem 3 gives sufficient conditions for the equicontinuity condition (i) and, therefore, (ii) and (iii), to hold.

As in the classical case we can apply the Uniform Boundedness Principle given in Theorem 3 to obtain a Banach-Steinhaus type result for order bounded operators ([1] 5.1). There is an order version of the Banach-Steinhaus Theorem for linear functionals due to Nakano given in [5] IX. 1.1, and a general form of the Banach-Steinhaus Theorem for operators between vector lattices given in [4]. The results in [4] treat a different class of operators than that considered below in Corollary 5 and the assumptions on the spaces are not as restrictive.

COROLLARY 5. Let X, Y be as in Theorem 3 and let $T_i : X \to Y$ be order bounded. If $0 - \lim T_i x = Tx$ exists for each $x \in X$, then $T : X \to Y$ is an order bounded linear operator.

PROOF. For each x, $\{T_i x\}$ is order bounded since the sequence is order convergent. Therefore, from Theorem 3, $\{T_i\}$ is uniformly order bounded on order bounded subsets of X. If $[u, v]$ is an order interval in X, then there is an order interval $[-w, w]$ in Y such that $T_i([u, v]) \subseteq [-w, w]$ for all i. Hence, $T([u, v]) \subseteq [-w, w]$ and T is order bounded.

The sequence of operators $\{T_i\}$ in Example 1 is pointwise order convergent to the operator $T : X \to Y$ given by $Tf = \sum_{j=1}^{\infty} f_j e_j$. However, T is not order bounded since $\{T\varphi_j\} = \{e_j/2\}$, where $\varphi_j(t) = \sin \pi jt$. This shows that in general the

Banach-Steinhaus result will not hold if the range space is merely Dedekind complete.

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REFERENCES

1. P. Antosik and C. Swartz, Matrix Methods in Analysis, Lecture Notes in Mathematics 1113, Springer-Verlag, 1985.

2. W.A.J. Luxemburg and A.C. Zaanen, Riesz Spaces I, North Holland, 1971.

3. A.L. Peressini, Ordered Topological Vector Spaces, Harper and Row, 1967.

4. C. Swartz, The Banach-Steinhaus Theorem for ordered spaces, Proceedings of the Conference GFAC, Dubrovnik, Yugoslavia, 1987, to appear.

5. B.Z. Vulikh, Introduction to the Theory of Partially Ordered Spaces, Wolters-Noordhoff, 1967.