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TWO CONSTRUCTIONS OF THE REAL NUMBERS VIA ALTERNATING SERIES

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ABSTRACT: Two further new methods are put forward for constructing the complete ordered field of real numbers out of the ordered field of rational numbers. The methods are motivated by some little known results on the representation of real numbers via alternating series of rational numbers. Amongst advantages of the methods are the facts that they do not require an arbitrary choice of "base" or equivalence classes or any similar constructs. The methods bear similarities to a method of construction due to Rieger, which utilises continued fractions.

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1. INTRODUCTION.

The series of Engel (1913) and Sylvester (1880) (see Perron [1]) for representing real numbers have been studied in some detail. Much less known is the fact that there are alternating series representations of real numbers in terms of rationals corresponding to the above. The only references to these alternating series that we are aware of in the literature are in papers of E Remez [2] and H Salzer [3]. The series under discussion are as follows: Every real number A has a unique representation in the form

$$A = a_0 + \frac{1}{a_1} - \frac{1}{a_1a_2} + \frac{1}{a_1a_2a_3} - \dots + \frac{(-1)^{n+1}}{a_1a_2\dots a_n} + \dots = (a_0,a_1,a_2,\dots).$$

say, where the a_i are integers such that $a_{i+1} \geqq a_i + 1 \geqq 2$ for $i \geqq 1$. Furthermore. A is *rational* if and only if A has a finite representation $(a_0,a_1,\dots, a_n)$. (Compare this with the expansion of Engel (Perron [1]).)

Corresponding to the series of Sylvester (Perron [1]) we have every real number

$$A = a_0 + \frac{1}{a_1} - \frac{1}{a_2} + \frac{1}{a_3} - \dots + \frac{(-1)^{n+1}}{a_n} + \dots = ((a_0,a_1,a_2,\dots)),$$

say, where the a_i are integers defined uniquely by A such that $a_1 \geqq 1$ and $a_{i+1} \geqq a_i(a_i + 1)$ for $i \geqq 1$. Furthermore, A is *rational* if and only if A has a finite representation $((a_0, a_1, \ldots, a_n))$.

In many ways, these representations may be compared with that by simple continued fractions. The main purpose of this note is to justify this remark by deriving some elementary properties of these alternating series representations and (with these results as an initial motivation) then developing two new methods for constructing the real number system from the ordered field of rational numbers. These methods are similar to one recently introduced by G J Rieger [4] for constructing the real numbers via continued fractions. The order relations in particular are defined in an analogous fashion.

The methods share with Rieger's method the advantage over other standard techniques that they do not require an arbitrary choice of a "base", or the use of (infinite) equivalence classes or similar such constructs. These properties are shared as well in the construction of real numbers using ordinary Sylvester and Engel series, considered in [5]. Two important differences between those and the present methods are in the definition of the order relations for the series, as well as the use here of terminating representations of rational numbers in place of infinite recurring representations used in [5].

2. ALTERNATING SERIES REPRESENTATIONS FOR REAL NUMBERS.

For the convenience of the reader, because full previous details may be inaccessible to many (including the present authors), we prove here the fundamental results concerning the representation of real numbers via infinite alternating series. It is convenient to introduce here a more general alternating series, analogous to the positive series of Oppenheim [6], out of which we can deduce the results for alternating-Engel and alternating-Sylvester series as special cases. We define the alternating-Oppenheim algorithm as follows:

Given any real number A, let $a_0 = [A]$, $A_1 = A - a_0$. Then we recursively define

$$a_n = \left[\frac{1}{A_n}\right] \geqq 1 \quad \text{for} \quad n \geqq 1,\ A_n > 0,$$

where

$$A_{n+1} = \left(\frac{1}{a_n} - A_n\right)(c_n/b_n) \quad \text{for} \quad a_n > 0.$$

Herein

$$b_i = b_i(a_1, a_2, \ldots, a_i) \quad , \qquad c_i = c_i(a_1, a_2, \ldots, a_i)$$

are positive numbers (usually integers).

The two cases of particular interest to us are those for which $b_n = 1$, $c_n = a_n$, $n \geqq 1$ (alternating-Engel series) and $b_n = c_n = 1, n \geqq 1$ (the alternating-Sylvester series).

THEOREM 2.1. *Every real number* A *has a unique representation in the form*

$$A = a_0 + \frac{1}{a_1} - \frac{1}{a_2} + \frac{1}{a_3} - \dots$$

where

$$a_{i+1} \geqq a_i(a_i + 1),\ a_1 \geqq 1 .$$

Furthermore every real number A *has a unique representation in the form*

$$A = a_0 + \frac{1}{a_1} - \frac{1}{a_1a_2} + \frac{1}{a_1a_2a_3} - \dots$$

where

$$a_{i+1} \geqq a_i + 1, a_1 \geqq 1.$$

PROOF. Repeated application of the alternating-Oppenheim algorithm yields

$$\begin{aligned} A &= a_0 + A_1 \\ &= a_0 + \frac{1}{a_1} - \frac{b_1}{c_1} a_2 = \dots \\ &= a_0 + \frac{1}{a_1} - \frac{b_1}{c_1} \cdot \frac{1}{a_2} + \frac{b_1b_2}{c_1c_2} \cdot \frac{1}{a_3} - \dots + (-1)^{k-1} \frac{b_1b_2\dots b_{k-1}}{c_1c_2\dots c_{k-1}} A_k . \end{aligned}$$

Now $\qquad a_n = [\frac{1}{A_n}]$ implies

$$\frac{1}{a_n+1} < A_n \leqq \frac{1}{a_n} \quad \text{for } 0 < A_n \leqq 1.$$

Thus

$$\begin{aligned} A_{n+1} &= (\frac{1}{a_n} - A_n)(c_n/b_n) \\ &< (\frac{1}{a_n} - \frac{1}{a_n+1})(c_n/b_n) \\ &= \frac{1}{a_n(a_n+1)} \cdot (c_n/b_n) , \quad \text{if } 0 < A_n \leqq 1 . \end{aligned}$$

In particular by setting $b_n = c_n = 1$ for all n we obtain

$$a_{n+1} = [\frac{1}{A_{n+1}}] \geqq a_n(a_n + 1) , \quad \text{if } A_i > 0 \text{ for } i \leqq n .$$

Furthermore $A_{n+1} < \frac{1}{a_n(a_n+1)} \to 0$ as $n \to \infty$, since $a_1 \geqq 1$ and the sequence $\{a_n\}$ is strictly increasing. It follows that A has an alternating-Sylvester expansion

$$A = a_0 + \frac{1}{a_1} - \frac{1}{a_2} + \frac{1}{a_3} - \dots \quad = ((a_0,a_1,a_2,\dots)) ,$$

which may perhaps terminate.

Secondly, by setting $c_n = a_n$, $b_n = 1$ for all n we obtain

$$a_{n+1} = [\frac{1}{A_{n+1}}] \geqq a_n + 1, \text{ if } A_i > 0 \text{ for } i \leqq n,$$

and so $\dfrac{A_{n+1}}{a_1 a_2 \ldots a_n} < \dfrac{\frac{1}{a_n+1}}{n!} \to 0$ as $n \to \infty$, since $a_1 \geqq 1$.

Thus A has the alternating-Engel expansion

$$A = a_0 + \frac{1}{a_1} - \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} - \ldots \quad = (a_0, a_1 . a_2, \ldots),$$

which also may terminate.

Uniqueness of the representations follows from Proposition 2.3 on order below. (Various other interesting special cases of the alternating-Oppenheim algorithm will be treated in a separate article.)

We deduce now an important result on the alternating series expansions for rational numbers.

PROPOSITION 2.2 *The alternating-Sylvester and alternating-Engel series terminate after a finite number of terms if and only if* A *is rational.*

PROOF. Clearly any number represented by a finite expansion is rational. Conversely, since $A_i, i \geqq 1$, is rational let $A_i = p_i/q_i$, $(p_i, q_i) = 1$. Now since for either algorithm

$$a_i = [\frac{1}{A_i}] > \frac{1}{A_i} - 1 \text{ it follows that } q_i - p_i a_i < p_i .$$

In the alternating-Sylvester case we now obtain

$$\frac{p_{i+1}}{q_{i+1}} = \frac{q_i - p_i a_i}{a_i q_i} .$$

Thus $0 \leqq p_{i+1} \leqq q_i - p_i a_i < p_i$. Since $\{p_i\}$ is a strictly decreasing sequence of non-negative integers we must eventually reach a stage at which $p_{n+1} = 0$, whence

$$A = a_0 + \frac{1}{a_1} - \frac{1}{a_2} + \ldots + \frac{(-1)^{n-1}}{a_n}$$

The result for the alternating-Engel series follows similarly from

$$\frac{p_{i+1}}{q_{i+1}} = \frac{q_i - p_i a_i}{q_i} .$$

We note that for rational numbers there is a possible ambiguity in the final term, analogous to that for continued fractions. We eliminate this as follows:

CONVENTION 1. We replace the finite sequence $((a_0, a_1, \ldots, a_n))$ by $((a_0, a_1, \ldots . a_{n-2} . a_{n-1}+1))$ in the case $a_n = a_{n-1}(a_{n-1}+1)$. Similarly we replace $(a_0, a_1, \ldots, a_n)$ by $(a_0, a_1, \ldots, a_{n-2}, a_{n-1}+1)$ in the case $a_n = a_{n-1}+1$. Furthermore, we identify $A \in \mathbb{Q}$ with its finite expansion $(a_0, a_1, \ldots, a_n)$ or $((a_0, a_1 \ldots . a_n))$, respectively.

In order to be able to compare finite sequences of different lengths in size we introduce the symbol ω with the following properties: For any $r \in \mathbb{Q}$, $r < \omega$, $\omega + r = \omega$, $\omega . \omega = \omega$.

Now we can represent finite sequences by infinite sequences as follows:

CONVENTION 2 For every $A = (a_0, a_1, \ldots, a_n) \in \mathbb{Q}$ let $a_j = \omega$ for $j > n$ and hence $A = (a_0, a_1, \ldots, a_n, \omega, \omega, \ldots)$. Similarly represent

$$A = ((a_0, a_1, \ldots, a_n)) \in \mathbb{Q} \quad \text{by} \quad A = ((a_0, a_1, \ldots, a_n, \omega, \omega, \ldots)).$$

PROPOSITION 2.3 (On Order) *Let* $A = (a_0, a_1, \ldots) \neq B = (b_0, b_1, \ldots)$, *or* $A = ((a_0, a_1, \ldots)) \neq B = ((b_0, b_1, \ldots))$. *In both of these cases, the condition* $A < B$ *is equivalent to:*

(i) $a_{2n} < b_{2n}$, *or*

(ii) $a_{2n+1} > b_{2n+1}$, *where* $i = 2n$ *or* $i = 2n+1$ *is the first index* $i \geq 0$ *such that* $a_i \neq b_i$.

PROOF. We shall use the notation $A_n' = \frac{1}{a_n} - \frac{1}{a_{n+1}} + \frac{1}{a_{n+2}} - \ldots$ for $A = ((a_0, a_1, a_2, \ldots))$, and $A_n' = \frac{1}{a_n} - \frac{1}{a_n a_{n+1}} + \frac{1}{a_n a_{n+1} a_{n+2}} - \ldots$ for $A = (a_0, a_1, a_2, \ldots)$. (Note that we do not assume at this stage that $A_n' = A_n$ as defined by either algorithm.)

Now suppose (i) holds. If firstly $a_0 < b_0$ then

$$A = a_0 + A_1 < a_0 + 1 \leq b_0 \leq b_0 + B_1 = B ,$$

in either case. Next suppose $a_{2n} < b_{2n}$, $n > 0$, in the alternating-Sylvester case. The growth condition

$$a_{i+1} \geq a_i(a_i + 1), \ i \geq 1,$$

implies that

$$\begin{aligned} A_{2n}' &= \frac{1}{a_{2n}} - \frac{1}{a_{2n+1}} + \frac{1}{a_{2n+2}} - \ldots \\ &\geq \frac{1}{a_{2n}} \left(1 - \frac{1}{a_{2n}+1}\right) + \frac{1}{a_{2n+2}} \left(1 - \frac{1}{a_{2n+2}+1}\right) + \ldots \\ &> \frac{1}{a_{2n}} \left(1 - \frac{1}{a_{2n}+1}\right) = \frac{1}{a_{2n}+1} , \end{aligned}$$

since $a_i > 1$ for $i > 1$, and by observing Convention 1. Furthermore,

$$\begin{aligned} A_{2n}' &= \frac{1}{a_{2n}} - \frac{1}{a_{2n+1}} + \frac{1}{a_{2n+2}} - \ldots \\ &\leq \frac{1}{a_{2n}} - \frac{1}{a_{2n+1}} \left(1 - \frac{1}{a_{2n+1}+1}\right) - \frac{1}{a_{2n+3}} \left(1 - \frac{1}{a_{2n+3}+1}\right) - \ldots \\ &\leq \frac{1}{a_{2n}} . \end{aligned}$$

Thus

$$A_{2n}' > \frac{1}{a_{2n}+1} \geq \frac{1}{b_{2n}} \geq B_{2n}' .$$

It now follows from $A = a_0 + \frac{1}{a_1} - \frac{1}{a_2} + \ldots - A_{2n}'$, $B = a_0 + \frac{1}{a_1} - \frac{1}{a_2} + \ldots - B_{2n}'$, that $A < B$. In the alternating-Engel case, from $a_{i+1} \geq a_i + 1$, $i \geq 1$,

$$A'_{2n} = \frac{1}{a_{2n}} - \frac{1}{a_{2n}a_{2n+1}} + \frac{1}{a_{2n}a_{2n+1}a_{2n+2}} - \dots$$

$$\geq \frac{1}{a_{2n}} \left(1 - \frac{1}{a_{2n}+1}\right) + \frac{1}{a_{2n}a_{2n+1}a_{2n+2}} \left(1 - \frac{1}{a_{2n+2}+1}\right) + \dots$$

$$> \frac{1}{a_{2n}} \left(1 - \frac{1}{a_{2n}+1}\right) = \frac{1}{a_{2n}+1} ,$$

as in the alternating-Sylvester case. Also

$$A'_{2n} = \frac{1}{a_{2n}} - \frac{1}{a_{2n}a_{2n+1}} + \frac{1}{a_{2n}a_{2n+1}a_{2n+2}} - \dots$$

$$\leq \frac{1}{a_{2n}} - \frac{1}{a_{2n}a_{2n+1}} \left(1 - \frac{1}{a_{2n+1}+1}\right) - \dots$$

$$\leq \frac{1}{a_{2n}} .$$

Thus again

$$A'_{2n} > \frac{1}{a_{2n}+1} \geq \frac{1}{b_{2n}} \geq B'_{2n} ,$$

and the result $A < B$ now follows from

$$A = a_0 + \frac{1}{a_1} - \frac{1}{a_1a_2} + \dots - \frac{1}{a_1a_2\cdots a_{2n-1}} A'_{2n}$$

$$B = a_0 + \frac{1}{a_1} - \frac{1}{a_1a_2} + \dots - \frac{1}{a_1a_2\cdots a_{2n-1}} B'_{2n} .$$

Note that if $b_{2n} = \omega$ then $B'_{2n} = 0$ and the result remains valid in this case. The result is proved in a similar fashion if (ii) holds.

3. CONSTRUCTIONS AND ORDER PROPERTIES

In the constructions below, standard facts about the ordered field $\mathbb{Q}$ of all rational numbers are taken as understood. With the results of Section 1 as initial motivation, we now define two sets $\mathcal{E}^*$ and $\mathcal{S}^*$ and order relations on them as follows:

Let $\mathcal{E}^*$ be the set of all formal infinite sequences $A = (a_0, a_1, a_2, \dots)$ of integers a_i such that $a_{i+1} \geq a_i + 1$ for $i \geq 1, a_1 \geq 1$. Also, let $\mathcal{S}^*$ be the set of all formal infinite sequences $A = ((a_0, a_1, a_2, \dots))$ of integers a_i such that $a_1 \geq 1$ and $a_{i+1} \geq a_i(a_i+1)$ for $i \geq 1$.

Finite sequences (rational numbers) are included in our sets $\mathcal{E}^*$ and $\mathcal{S}^*$ using Convention 2. We will frequently make use of the property all sequences in $\mathcal{E}^*$ and $\mathcal{S}^*$ satisfy:

$$a_i = \omega \text{ implies } a_j = \omega \text{ for all } j > i .$$

In both the sets $\mathcal{E}^*$ and $\mathcal{S}^*$ we shall use corresponding lower-case letters to denote the "digits" of the elements of the respective sets, and we define

$A < B$ if and only if

(i) $a_{2n} < b_{2n}$, or

(ii) $a_{2n+1} > b_{2n+1}$,

where $i = 2n$ or $i = 2n+1$ is the first index $i \geqq 0$ such that $a_i \neq b_i$.

LEMMA 3.1. *In both cases, $<$ is a "total ordering" relation, i.e., it is transitive and satisfies the trichotomy law.*

PROOF. We use the same argument in both cases. Firstly, trichotomy is obvious. Next let $A < B$ and $B < C$. Suppose $a_r = b_r$ for $r < i$, $a_i \neq b_i$, and $b_r = c_r$ for $r < j, b_j \neq c_j$.

(i) If $i < j$ then $a_r = c_r$ for $r < i$, and $a_i < b_i = c_i$ (i even) or $a_i > b_i = c_i$ (i odd).

(ii) If $i = j$ then $a_r = c_r$ for $r < j$, and $a_i < b_i < c_i$ (i even) or $a_i > b_i > c_i$ (i odd).

(iii) If $i > j$ then $a_r = c_r$ for $r < j$, and $a_j = b_j < c_j$ (j even) or $a_j = b_j > c_j$ (j odd). Thus $A < C$ in each case.

We may now introduce symbols $\leqq$, $>$ and $\geqq$, and define (*least*) *upper bounds* and (*greatest*) *lower bounds*, in the usual way.

LEMMA 3.2. *Every non-empty subset of $\mathcal{E}^*$ (respectively, $\mathcal{S}^*$) which is bounded above has a least upper bound* (supremum).

PROOF. First consider a non-empty subset X of $\mathcal{E}^*$, which is bounded above by a sequence $B = (b_0, b_1, \ldots)$.

Assume $B \notin X$, since otherwise there is nothing to prove. Now $A < B$ for every $A \in X$, and there is a largest index k such that every $A \in X$ with $a_0 = b_0$ has $a_1 = b_1, \ldots, a_k = b_k$. We may assume $a_0 = b_0$ for some $A \in X$ since otherwise $(d_0, 1, \omega, \omega, \ldots)$ is an upper bound for X, where d_0 is the maximum value of a_0 for elements of X.

We now define $c_0 = b_0, \ldots, c_k = b_k$. If $k+1$ is odd let c_{k+1} be the least possible value for the digit a_{k+1} of any $A \in X$ with $a_0 = b_0$. If $k+1$ is even let c_{k+1} be the greatest possible value for the digit a_{k+1} of any $A \in X$ with $a_0 = b_0$, where we take $c_{k+1} = \omega$ if a_{k+1} has no largest value. In either case if $c_{k+1} = \omega$ we are done, and put $C = (c_0, c_1, \ldots, c_k, \omega, \omega, \ldots,)$. Otherwise we continue to define c_{k+2} as the least possible value or greatest possible value depending on whether $k+2$ is odd or even, respectively, for the digit a_{k+2} of all elements of the form $(c_0, c_1, \ldots, c_{k+1}, a_{k+2}, a_{k+3}, \ldots)$ in X. Again if $c_{k+2} = \omega$ we are done and put $C = (c_0, c_1, \ldots, c_{k+1}, \omega, \omega, \ldots.)$. Continue inductively, to define c_{k+i+1} as the least possible value ($k+i+1$ odd) or greatest possible value ($k+i+1$ even) for the digit a_{k+i+1} of an element of X of the form $(c_0, c_1, \ldots, c_{k+i}, a_{k+i+1}, a_{k+i+2}, \ldots)$. If, when $k+i+1$ is even, a_{k+i+1} has no largest value, we take $c_{k+i+1} = \omega$. The process terminates if at any stage $c_{k+i+1} = \omega$. We then take $C = (c_0, c_1, \ldots, c_{k+i}, \omega, \omega, \ldots)$. Otherwise this process constructs a non-terminating sequence $C = (c_0, c_1, \ldots)$. In either case we have $C \in E^*$ since $c_{i+1} \geq c_i + 1$ for $i \geq 1, c_1 \geq 1$. Also if $C \neq A \in X$ then $C > A$ since either $c_i > a_i$ (i even) or $c_i < a_i$ (i odd) for the first index $i > k$ such that $c_i \neq a_i$ (by the definition of the sequence C).

Lastly, $C = \sup X$ since otherwise X has an upper bound $D < C$. Then $d_i = c_i, 0 \leq i < m, d_m \neq c_m$. If m is odd then $d_m > c_m$. Hence every element of the form $A = (c_0, c_1, \ldots, c_m, a_{m+1}, a_{m+2}, \ldots)$ in X satisfies $D < A \leq D$; contradiction. If m is even we have $d_m < c_m$. In the cases $m = 0$ or $c_m < \omega$ $(m > 0)$ every element of the form $A = (c_0, c_1, \ldots, c_m, a_{m+1}, a_{m+2}, \ldots) \in X$ satisfies $D < A \leq D$. In the case $c_m = \omega, a_m \neq \omega$, for every $A \in X$ we can choose $A = (c_0, c_1, \ldots, c_{m-1}, a_m, a_{m+1}, \ldots) \in X$ with arbitrarily large a_m. For $a_m > d_m$ we have $D < A \leq D$. Finally, if $c_m = \omega$ and $a_m = \omega$ for some $A \in X$, choose $A = (c_0, c_1, \ldots c_{m-1}, \omega, \omega, \ldots)$ and $D < A \leq D$ once more.

The argument for S^* is almost identical to the above, except that the sequence $C = ((c_0, c_1, c_2, \ldots))$ defined inductively via suitable elements of S^* will now satisfy $c_{i+1} \geq c_i(c_i+1), c_1 \geq 1$.

4. EMBEDDING AND DENSITY OF RATIONALS

The following proposition justifies our use of Convention 1.

PROPOSITION 4.1 *The alternating-Engel and alternating-Sylvester algorithms define* 1 - 1 *order-preserving maps*

$$\rho_{E^*} : \mathbf{Q} \to E^* \quad \text{and} \quad \rho_{S^*} : \mathbf{Q} \to S^* ,$$

whose images are dense in E^* *and* S^*, *respectively*.

PROOF. It is an immediate consequence of the results quoted earlier that the two algorithms define 1 - 1 maps $\rho_{E^*} : \mathbf{Q} \to E^*$ and $\rho_{S^*} : \mathbf{Q} \to S^*$. By Proposition 2.3 and the definition of order in E^* and S^*, these maps are then order-preserving.

Now let $A < B$ in E^*. Let k be the least index for which $a_k \neq b_k$. We show now in every possible case that we can find a rational number D satisfying $A < D < B$. For $A \in \mathbf{Q}$, $B \in \mathbf{Q}$ we take $D = \frac{A+B}{2}$. Now let $A \notin \mathbf{Q}$ or $B \notin \mathbf{Q}$. If k is even then $a_k < b_k$; we choose $D = (b_0, b_1, \ldots b_k, b_{k+1}+1, \omega, \omega, \ldots)$ if $b_{k+1} \neq \omega$, or $D = (a_0, a_1, \ldots, a_k, a_{k+1}, a_{k+2}+1, \omega, \omega, \ldots)$ if $b_{k+1} = \omega$, i.e., $B \in \mathbf{Q}, A \notin \mathbf{Q}$. If instead k is odd then $a_k > b_k$; we choose $D = (a_0, a_1, \ldots a_k, a_{k+1}+1, \omega, \omega, \ldots)$ if $a_{k+1} \neq \omega$, or $D = (b_0, b_1, \ldots, b_k, b_{k+1}, b_{k+2}+1, \omega, \omega, \ldots)$ if $a_{k+1} = \omega$ and $A \in \mathbf{Q}, B \notin \mathbf{Q}$. A similar argument works in the case of S^*

We note that sequences in E^* and S^* have the intuitively desirable property that rational numbers can be represented only by finite sequences (excluding ω's) This does not hold in the case of ordinary Engel and Sylvester series (see for example [1]).

APPROXIMATION LEMMA 4.2 *Given any element* A *of* $\mathcal{E}^*$ *(respectively,* $\mathcal{S}^*$*), there exist rationals* $A^{(n)}$ *for* $n \geqq 0$ *such that*

(i) $A^{(2m)} \leqq A^{(2n)} \leqq A \leqq A^{(2n+1)} \leqq A^{(2m+1)}$ *for* $m < n$,

(ii) $A = \sup A^{(2n)} = \inf A^{(2n+1)}$,

(iii) $A^{(2n+1)} - A^{(2n)} \leqq \dfrac{1}{(2n+1)!}$.

PROOF. Given $A = (a_0, a_1, \ldots) \in \mathcal{E}^*$, define the rational sequences $A^{(n)}$ by $A^{(n)} = (a_0, a_1, \ldots, a_n, \omega, \omega, \ldots)$. Then part(i) follows. Next suppose that $A < B \leqq A^{(2n+1)}$ for all n. In that case, we must have $a_m > b_m$ if m is odd, or $a_m < b_m$ if m is even, for the first index m such that $a_m \neq b_m$. Then m odd gives the contradiction $A^{(m)} < B \leqq A^{(m)}$, while m even gives the contradiction $A^{(m+1)} < B \leqq A^{(m+1)}$. Thus $A = \inf A^{(2n+1)}$. Similarly, suppose that $A^{(2n)} \leqq C < A$ for all n . Consider the first index m for which $a_m \neq c_m$. If m is even we must have $a_m > c_m$, which yields the contradiction $A^{(m)} \leqq C < A^{(m)}$. If m is odd we have $a_m < c_m$, which gives the contradiction $A^{(m+1)} \leqq C < A^{(m+1)}$. The same argument leads to parts (i) and (ii) for $\mathcal{S}^*$.

For part (iii) in $\mathcal{E}^*$, the formula for alternating-Engel series for rationals leads to

$$A^{(2n+1)} - A^{(2n)} = \frac{1}{a_1 a_2 \cdots a_{2n+1}} \leqq \frac{1}{(2n+1)!} ,$$

since $a_1 \geqq 1, a_{i+1} \geqq a_i + 1$. For $\mathcal{S}^*$, the corresponding formula for alternating-Sylvester series of rationals gives instead

$$A^{(2n+1)} - A^{(2n)} = \frac{1}{a_{2n+1}} < \frac{1}{(2n+1)!} ,$$

since $a_1 \geqq 1$ and $a_{i+1} \geqq a_i(a_i+1)$.

5. ALGEBRAIC OPERATIONS IN $\mathcal{E}^*$ AND $\mathcal{S}^*$

Since we alreaady regard $\mathbb{Q}$ as an actual subset of $\mathcal{E}^*$ and $\mathcal{S}^*$ by Convention 1, it will simplify the discussion on algebraic operations below if we now *re-define*

$$A^{(n)} = A \quad (n \geqq 0) ,$$

for any *rational* A.

For any $A, B \in \mathcal{E}^*$ (or $A, B \in \mathcal{S}^*$) we now define $A + B = \sup(A^{(2n)} + B^{(2n)})$, $-A = \sup(-A^{(2n+1)})$, which exist in $\mathcal{E}^*$ (respectively, $\mathcal{S}^*$) because

$$A^{(2n)} + B^{(2n)} \leqq A^{(1)} + B^{(1)} , \ -A^{(2n+1)} \leqq -A^{(0)} .$$

At this stage we note that the formal structures of the sets $\mathcal{E}^*$ and $\mathcal{S}^*$ are very similar to the set K, (based on continued fractions) used by Rieger [4] to construct the real numbers. Thus to avoid repetition, we will refer the reader to the corresponding result of Rieger whenever the proof of the algebraic property there is the same.

LEMMA 5.1 *The above operations make* $\mathcal{E}^*$ *(respectively,* $\mathcal{S}^*$*) into an abelian group containing* $(\mathbb{Q}, +)$ *as a dense subgroup. Further*

(i) $A < B \Rightarrow A + C < B + C$,

(ii) $A < B \Rightarrow -A > -B$.

PROOF. Obviously $A + B = B + A$ and $A + 0 = A$. The proof of the associative law of addition can be found in Theorem 3.1 of [4]. Also the proof that $A + (-A) = 0$ is found in Theorem 3.2 of [4]. The only changes that must be made are that we use $A^{(2n+1)} - A^{(2n)} \leqq \frac{1}{(2n+1)!}$ in place of inequality (1.4) of [4].

It now follows that $\mathcal{E}^*$ (respectively, $\mathcal{S}^*$) forms an abelian group with $(\mathbb{Q},+)$ as a dense subgroup. Then $A + C = B + C = A = B$ and hence the strict monotone law (i) follows from the weak one proved below:

By the definitions of order and rational approximations we have $A < B \Rightarrow A^{(2n)} < B^{(2n)}, A^{(2n+1)} < B^{(2n+1)}$, for n sufficiently large. Hence $A < B \Rightarrow A + C \leqq B + C$.

Finally, let $A < B$. Then $A^{(2n+1)} < B^{(2n+1)}$ or $-A^{(2n+1)} > -B^{(2n+1)}$, for n sufficiently large. Thus

$-A = \sup(-A^{(2n+1)}) \geqq \sup(-B^{(2n+1)}) = -B$, giving $-A > -B$ since $A \neq B$.
Hence (ii) follows, since $-(-X) = X$.

Next, for any $A,B \in \mathcal{E}^*$ (respectively, $\mathcal{S}^*$), define

$$A.B = \begin{cases} \sup(A^{(2n)}.B^{(2n)}) & \text{if } A \geqq 0, B \geqq 0 , \\ (-A).(-B) & \text{if } A \leqq 0, B \leqq 0 , \\ -((-A).B) & \text{if } A \leqq 0, B \geqq 0 , \\ -(A.(-B)) & \text{if } A \geqq 0, B \leqq 0 . \end{cases}$$

Also define

$$A^{-1} = \begin{cases} \sup((A^{(2n+1)})^{-1}) & \text{if } A > 0 , \\ -((-A)^{-1}) & \text{if } A < 0 . \end{cases}$$

The definitions are unambiguous, since we have

$$A^{(2n)} . B^{(2n)} \leqq A^{(1)} .B^{(1)} , \quad (A^{(2n+1)})^{-1} \leqq (A^{(0)})^{-1}$$

for $A > 0, B > 0$ and the theorem of the supremum is applicable. To cover all the cases, we use the fact that $A < 0$ if and only if $-A > 0$, by Lemma 5.1(ii).

LEMMA 5.2 *The above definitions together with the earlier operations make* $\mathcal{E}^*$ *(respectively,* $\mathcal{S}^*$*) into a field containing* $\mathbb{Q}$ *as a dense subfield,* *Further,*

$$A < B, C > 0 \Rightarrow A.C < B.C .$$

PROOF. Clearly $A.B = B.A$ and $A.1 = A$. Also, since $C > 0$ implies $C^{(n)} > 0$ for all n, we obtain easily

$$0 < A < B,\ C > 0 \Rightarrow A.C \leqq B.C$$

(with strict inequality to be shown later).
In order to verify that $\mathcal{E}^*(\mathcal{S}^*)$ is a field it remains only to verify that . is associative, and distributive relative to +, and that $A^{-1} .A = 1, A \neq 0$. The

associative law appears in Theorem 4.1 of [4], the distributive law in Theorem 4.2 of [4], and $A^{-1}.A = 1$, $A \neq 0$ is shown in Theorem 5.1 of [4]. In each case we must replace Rieger's inequality (1.4) by

$$A^{(2n+1)} - A^{(2n)} \leqq \frac{1}{(2n+1)!} .$$

The strict monotone law for multiplication of positive elements now follows from the weak one, and the law $A.C = B.C \Rightarrow A = B$ (for positive elements). By simple manipulation of the "sign" cases the result follows for all elements in $\mathcal{E}^*$ (or $\mathcal{S}^*$).

The above discussion has shown that both $\mathcal{E}^*$ and $\mathcal{S}^*$ form ordered fields with the least upper bound property. By standard theorems, treated for example in Chapter 5 of Cohen and Ehrlich [7], it then follows that $\mathcal{E}^*$ and $\mathcal{S}^*$ form concrete new models for the real number system $\mathbb{R}$. The models are in many ways equivalent to that of Rieger [4], except that they arise from the (simpler) representation of real numbers as infinite series rather than as infinite continued fractions.

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