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A TOPOLOGICAL LATTICE ON THE SET OF MULTIFUNCTIONS

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ABSTRACT. Let X be a Wilker space and M(X,Y) the set of continuous multifunctions from X to a topological space Y equipped with the compact-open topology. Assuming that M(X,Y) is equipped with the partial order $\subset$, we prove that $(M(X,Y),\subset)$ is a topological $\vee$-semilattice. We also prove that if X is a Wilker normal space and U(X,Y) is the set of point-closed upper semi-continuous multifunctions equipped with the compact-open topology, then $(U(X,Y),\subset)$ is a topological lattice.

KEY WORDS AND PHRASES. Continuous multifunctions, upper semicontinuous multifunctions, compact-open topology, topological lattice.
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1. INTRODUCTION AND DEFINITIONS.

A mapping F from a set X to a set Y which maps each point of X to a subset of Y is called multifunction. For any subset A of X, $F(A) = \bigcup_{x \in A} F(x)$. For any subset B of Y, $F^{+}(B) = \{x \in X : F(x) \subset B\}$ and $F^{-}(B) = \{x \in X : F(x) \cap B \neq 0\}$. Let X and Y be topological spaces.

A multifunction F from X to Y is upper semi-continuous (lower semi-continuous) if and only if $F^{+}(P)$ $(F^{-}(P))$ is open for each open subset P of Y (see Smithson [1]).

A multifunction $F: X \to Y$ is continuous if and only if it is both upper and lower semi-continuous [1].

A multifunction $F: X \to Y$ is point-closed [1] if and only if F(x) is a closed subset of Y, for each $x \in X$.

If F_1, F_2 are two multifunctions from X to Y, by $F_1 \vee F_2$, we denote the multifunction from X to Y defined by $(F_1 \vee F_2)(x) = F_1(x) \cup F_2(x)$. Also, by $F_1 \wedge F_2$, we denote the multifunction from X to Y defined by $(F_1 \wedge F_2)(x) = F_1(x) \cap F_2(x)$ in Kuratowski [2].

In the following, by M(X,Y), we denote the set of continuous multifunctions. Also, by U(X,Y), we denote the set of point-closed upper semi-continuous multifunctions.

Let K be a compact subset of X and P an open subset of Y. Let <K,P> = $\{F \in M(X,Y): F(x) \cap P \neq 0$ for all $x \in K\}$ and $[K,P] = \{F \in M(X,Y): F(K) \subset P\}$. The topology T_{co} on M(Y,Z) generated by the sets of the form <K,P> and [K,P], where K is compact in X and P is open in Y, is called the compact open topology on M(X,Y) [1].

The topology T^*_{co} on U(X,Y) generated by the sets of the form $[K,P] = \{F \in U(X,Y): F(K) \subset P\}$, where K is compact in X and P open in Y, is called the compact-open topology on U(X,Y).

For simplicity, in what follows, we use the symbols M(X,Y) (U(X,Y)) to denote the topological spaces $(M(X,Y), T_{co})$ $((U(X,Y), T^*_{co}))$.

We give now the definition of Wilker spaces that we will use in the following: A topological space X satisfies the Wilker's condition (D) For every compact subset $K \subset X$ and for every pair of open subsets $A_1, A_2 \in X$ with $k \subset A_1 \cup A_2$ there are compact subsets $K_1 \subset A_1$ and $K_2 \subset A_2$ such that $K \subset K_1 \cup K_2$ is called a Wilker space (Wilker [3]). It can be easily proved that the class of Wilker spaces contains properly the class of T_2 spaces and also the class of basic locally compact spaces (i.e. those spaces every point of which has a neighborhood basis consisting of compact sets). In [4] basic locally compact spaces are called locally quasi-compact spaces and in Murdehswar [5] they are called spaces which satisfy condition L_2.

In this paper we prove that if X is a Wilker space, then the $\vee$-semilattices $(M(X,Y), \subset)$, $(U(X,Y), \subset)$ are topological, i.e., we prove the continuity of the join operation $\vee$. It is also noticed that if X is a normal space, $(U(X,Y), \subset)$ is a semilattice [4,p.4]. Finally, if X is a Wilker normal space, we prove that the meet operation $\wedge$ is continuous, i.e., $(U(X,Y), \subset)$ is a topological semilattice [4, p.274].

The worth of the above results relies on the fact that the space U(X,Y) (M(X,Y)) can be considered as a topological lattice (topological $\vee$-semilattice [4,p.4]).

2. MAIN RESULTS.

PROPOSITION 2.1. Let X be a Wilker space. Then the operation $(F_1,F_2) \to F_1 \vee F_2 : M(X,Y) \times M(X,Y) \to M(X,Y)$ is continuous. Thus the $\vee$-semilattice $(M(X,Y), \subset)$ is topological.

PROOF. Let $(F_1,F_2) \in M(X,Y) \times M(X,Y)$ and $F_1 \vee F_2 \in [K,P]$. Then $(F_1 \vee F_2)(K) \subset P$, which implies that $F_1(K) \subset P$ and $F_2(K) \subset P$. Hence $F_1 \in [K,P]$, $F_2 \in [K,P]$ and it can be easily proved that $(G_1,G_2) \in [K,P] \times [K,P]$ implies that $G_1 \vee G_2 \in [K,P]$.

Let now $F_1 \vee F_2 \in \langle K,P\rangle$. Then $(F_1 \vee F_2)(x) \cap P \neq 0$ for each $x \in K$. So we have $K \subset F_1^-(P) \cup F_2^-(P)$. But since X is a Wilker space there are compact subsets K_1, K_2 of X, such that $K_i \subset F_i^-(P)$, $i = 1, 2$, and $K \subset K_1 \cup K_2$. So $F_1 \in \langle K_1, P\rangle$, $F_2 \in \langle K_2, P\rangle$. We prove now that $(G_1, G_2) \in \langle K_1,P\rangle \times \langle K_2, P\rangle$ implies that $G_1 \vee G_2 \in \langle K, P\rangle$.
Let $(G_1, G_2) \in \langle K_1, P\rangle \times \langle K_2, P\rangle$. Then, $K_i \subset G_i^-(P)$, $i = 1, 2$, which implies that $K \subset K_1 \cup K_2 \subset G_1^-(P) \cup G_2^-(P) = (G_1 \vee G_2)^-(P)$. Therefore $G_1 \vee G_2 \in \langle K, P\rangle$.

The proof of the following Proposition is the same as that of Proposition 2.1 (first part) and it is omitted.

PROPOSITION 2.3. Let X be a Wilker space. Then the operation $(F_1,F_2) \to F_1 \vee F_2 \colon U(X,Y) \times U(X,Y) \to U(X,Y)$ is continuous. Thus, the $\vee$-semilattice $(U(X,Y), \subset)$ is topological.

LEMMA 2.3. [2, p.179]. Suppose X is a normal space. Let $F_1 \colon X \to Y$, $F_2 \colon X \to Y$ be two point-closed upper semi-continuous multifunctions and P an open set in Y. Then,

$$(F_1 \wedge F_2)^+(P) = \cup\{F_1^+(V) \cap F_2^+(W)\}, \text{ where } V,W \text{ are open in } Y,\ V \cap W = P.$$

PROPOSITION 2.4. Consider a Wilker normal space X. Let U(X,Y) be the set of point closed upper semi-continuous multifunctions equipped with the compact-open topology. Then $(U(X,Y), \subset)$ is a topological lattice.

PROOF. It suffices to prove that $(U(X,Y), \subset)$ is a topological similattice, i.e., that the meet operation $\wedge$ is continuous. According to the previous lemma, it is obvious that the function $(F_1,F_2) \to F_1 \;\; F_2 \colon U(X,Y) \times U(X,Y) \to U(X,Y)$ is well defined, i.e. that $(U(X,Y), \subset)$ is a semilattice.

We prove now that $\wedge$ continuous.

Let an arbitrary $(F_1,F_2) \in U(X,Y) \times U(X,Y)$ and let $F_1 \wedge F_2 \in [K,P]$, where K is compact in X and P is open in Y. Then by the previous lemma

$$K \subset (F_1 \wedge F_2)^+(P) = \cup\{F_1^+(V) \cap F_2^+(W)\},$$

where V,W are open in Y, $V \cap W = P$. But since K is compact there are finitely many sets V_i, W_i, $i = 1,\dots,n$ such that

$$K \subset \bigcup_{i=1}^{n} \{F_1^+(V_i) \cap F_2^+(W_i)\},$$

where V_i, W_i, are open in Y, $V_i \cap W_i = P$, $i = 1,\dots, n$. Moreover since X is a Wilker space there exist compact subsets of X, K_i, $i=1,\dots,n$, such that

$$K_i \subset F_1^+(V_i) \cap F_2^+(W_i) \text{ and } K \subset \bigcup_{i=1}^{n} K_i.$$

Thus, $K_i \subset F_1^+(V_i)$, $K_i \subset F_2^+(W_i)$, $i = 1,\dots, n$.

Hence $F_1 \in [K_i,V_i]$, $F_2 \in [K_i, W_i]$, $i = 1,\dots, n$ and finally

$$(F_1,F_2) \in \bigcap_{i=1}^{n} [K_i, V_i] \times \bigcap_{i=1}^{n} [K_i, W_i].$$

It remains to prove that for each

$$(G_1, G_2) \in \bigcap_{i=1}^{n} [K_i, V_i] \times \bigcap_{i=1}^{n} [K_i, W_i],\ G_1 \wedge G_2 \in [K, P].$$

To prove this consider an aribtrary

$$(G_1, G_2) \in \bigcap_{i=1}^{n} [K_i, V_i] \times \bigcap_{i=1}^{n} [K_i, W_i].$$

It must be shown that $K \quad (G_1 \quad G_2)^+(P)$. Let an arbitrary $x \in K$. Then $x \in K_i$, for some i, $1 < i < n$. Since $K_i \subset G_1^+(V_i)$, $K_i \subset G_2^+(W_i)$, we have that $G_1(x) \subset V_i$, $G_2(x) \subset W_i$. So $G_1(x) \cap G_2(x) = (G_1 \wedge G_2)(x) \quad V_i \cap W_i = P$. Thus, $x \in (G_1 \wedge G_2)^+(P)$, which completes the proof.

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