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SUFFICIENT CONDITIONS FOR SPIRAL-LIKENESS

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ABSTRACT. Coefficient conditions sufficient for spiral-likenss are found by convolution methods. The order of starlikeness for such functions is also determined.

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1. INTRODUCTION.

A function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic in the unit disk $\Delta = \{|z| < 1\}$ is said to be starlike of order α, $0 \leq \alpha < 1$, if $\text{Re}\,\{zf'/f\} > \alpha$, $z \in \Delta$, and is said to be λ spiral-like, $-\pi/2 < \lambda < \pi/2$, if $\text{Re}\{e^{i\lambda} zf'/f\} > 0$, $z \in \Delta$. We denote these classes, respectively, by $S^*(\alpha)$ and $Sp(\lambda)$. Note that $Sp(0) = S^*(0)$, the family of starlike functions. Functions in $Sp(\lambda)$ were shown by Spacek [4] to be univalent in Δ and were later studied extensively by Libera [1].

A function f is in $S^*(\alpha)$ if its coefficients are sufficiently small.

THEOREM A [2]. *If the coefficients of* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *satisfy the inequality*

$$(1) \qquad \sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha} |a_n| \leq 1,$$

then $|(zf'/f) - 1| < 1-\alpha$, $z \in \Delta$, *and hence* $f \in S^*(\alpha)$.

It is our purpose here to find coefficient conditions guaranteeing that f is in $Sp(\lambda)$. Our methods will involve convolution properties and will also furnish us with an alternate proof that (1) is a sufficient condition for f to be in $S^*(\alpha)$.

The *convolution* or *Hadamard product* of two power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined as the power series $(f*g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n$. In [3] it is shown that $f \in S^*(\alpha)$ if and only if

$$\frac{1}{z}\left(f* \frac{z + ((x+2\alpha-1)/(2-2\alpha))z^2}{(1-z)^2}\right) \neq 0 \qquad (z \in \Delta,\ |x| = 1)$$

and $f \in Sp(\lambda)$ if and only if

$$\frac{1}{z}\left(f* \frac{z + ((x-e^{-2i\lambda})/(1+e^{-2i\lambda}))z^2}{(1-z)^2}\right) \neq 0 \quad (z \in \Delta,\ |x| = 1).$$

Now $(z + cz^2)/(1-z)^2 = z + \sum_{n=2}^{\infty} (n + (n-1)c)z^n$, so we may restate these results as

THEOREM B. The function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*(\alpha)$ $(Sp(\lambda))$ <u>if and only if</u>

$1 + \sum_{n=2}^{\infty} (n + (n-1)c)a_n z^{n-1} \neq 0$ <u>for all</u> $z \in \Delta$ <u>and</u> $|x| = 1$, <u>where</u>

$$c = \frac{x+2\alpha-1}{2-2\alpha} \quad \left(c = \frac{x-e^{-2i\lambda}}{1+e^{-2i\lambda}}\right).$$

Since $\left|1 + \sum_{n=2}^{\infty} (n + (n-1)c)a_n z^{n-1}\right| > 1 - \sum_{n=2}^{\infty} |n + (n-1)c||a_n||z|^{n-1}$, a sufficient condition for f to be in $S^*(\alpha)$ or $Sp(\lambda)$ is that $\sum_{n=2}^{\infty}|n + (n-1)c||a_n| < 1$ for the appropriate choice of c. A straightforward computation shows that

$$\left|n + \frac{(n-1)(x+2\alpha-1)}{2-2\alpha}\right| < \left|n + \frac{(n-1)(2\alpha)}{2-2\alpha}\right| = \frac{n-\alpha}{1-\alpha},$$

and we can conclude from Theorem B that condition (1) is sufficient for f to be in $S^*(\alpha)$. The corresponding result for $Sp(\lambda)$ is computationally more involved.

2. THE MAIN CLASS.

THEOREM 1. <u>The function</u> $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in Sp(\lambda)$ <u>if</u> $\sum_{n=2}^{\infty} B_n(\lambda)|a_n| < 1$ <u>for</u>

$$B_n(\lambda) = \frac{(n-1) + \sqrt{(n-1)^2 + 4n\cos^2\lambda}}{2\cos\lambda}.$$

<u>The result is sharp, with</u> $f_n(z) = z + a_n z^n$ <u>in</u> $Sp(\lambda)$ <u>if and only if</u> $|a_n| < 1/B_n(\lambda)$.

PROOF. From Theorem B, it suffices to show for $c = (x-e^{-2i\lambda})/(1+e^{-2i\lambda})$ that

$\max_{|x|=1} |n + (n-1)c| = B_n(\lambda)$. Writing $c = c_1 + ic_2$, c_1 and c_2 real, we have

$$(2) \qquad |n + (n-1)c| = \sqrt{n^2 + (n-1)[(n-1)(c_1^2 + c_2^2) + 2nc_1]}\,.$$

Hence $|n + (n-1)c|$ will attain its maximum when $(n-1)(c_1^2 + c_2^2) + 2nc_1$ does. Setting $x = e^{i\beta}$, a computation shows that

$$c_1^2 + c_2^2 = \frac{1 - \cos(2\lambda + \beta)}{1 + \cos 2\lambda} \text{ and } c_1 = \frac{\cos(2\lambda + \beta) + \cos\beta - (1 + \cos 2\lambda)}{2(1 + \cos 2\lambda)}.$$

Thus

$$(n-1)(c_1^2 + c_2^2) + 2nc_1 = \frac{\cos(2\lambda + \beta) + n\cos\beta - (1 + n\cos 2\lambda)}{2(1 + \cos 2\lambda)},$$

which is maximized when $g(\beta) = \cos(2\lambda + \beta) + n\cos\beta$ is maximized. But $g(\beta)$ attains its maximum when $\beta = \beta_0 = \tan^{-1}\left(-\frac{\sin 2\lambda}{n + \cos 2\lambda}\right)$, with $g(\beta_0) = \sqrt{(n-1)^2 + 4n\cos^2\lambda}$. For this choice of β_0, we have

$$(n-1)(c_1^2 + c_2^2) + 2nc_1 = \frac{\sqrt{(n-1)^2 + 4n\cos^2\lambda} + (n-1) - 2n\cos^2\lambda}{2\cos^2\lambda} = t_n(\lambda).$$

It now follows from (2) that

$$\max_{|x|=1} |n + (n-1)c| = (n^2 + (n-1)t_n(\lambda))^{1/2},$$
$$= \left(\frac{(n-1)^2 + 2n\cos^2\lambda + (n-1)\sqrt{(n-1)^2 + 4n\cos^2\lambda}}{2\cos^2\lambda}\right)^{1/2}.$$

which may be expressed as $B_n(\lambda)$. This completes the proof. To show sharpness, note that according to Theorem B $f_n(z) = z + a_n z^n \notin Sp(\lambda)$ if $z^{n-1} = -[(n + (n-1)c)a_n]^{-1}$ has a solution for $z \in \Delta$. Choosing c so that $|n + (n-1)c| = B_n(\lambda)$, we see that $f_n(z) \notin Sp(\lambda)$ if $|a_n| > 1/B_n(\lambda)$. In particular, $f_n(z) = z + a_n z^n \in Sp(\lambda)$ if and only if $|a_n| \le 1/B_n(\lambda)$.

COROLLARY 1. <u>If</u> $f_n(z) = z + a_n z^n \in Sp(\lambda_0)$, <u>then</u> $f_n \in Sp(\lambda)$ <u>for</u> $|\lambda| < |\lambda_0|$.

PROOF. This is a consequence of $B_n(\lambda)$ being an increasing function of $|\lambda|$.

In fact, any function that satisfies the conditions of Theorem 1 for $\lambda = \lambda_0$ will also be in $Sp(\lambda)$ for $|\lambda| < |\lambda_0|$, a sharp contrast to the inclusion properties for the general class $Sp(\lambda)$. The function

$f_\lambda(z) = z(1-z)^{-2e^{-i\lambda}\cos\lambda}$ is in $Sp(\lambda)$ but it is not in $Sp(\lambda)$ for any $\gamma \neq \lambda$. On the other hand, the upper bound on the modulus of the coefficients for $f \in Sp(\lambda)$ is a decreasing function of $|\lambda|$. Zamorski [5] showed the sharp coefficient bounds $|a_n|$ for $f \in Sp(\lambda)$ to be

$$|a_n| = \prod_{k=1}^{n-1} \sqrt{(k-1)^2 + 4k\cos^2\lambda}/(n-1)!,$$ with $f_\lambda(z)$ being extremal.

Though Theorem 1 gives a sharp result, it is not aesthetically pleasing because of the complicated nature of $B_n(\lambda)$. A consequence of the inequality $1 + (n-1)\sec\lambda > B_n(\lambda)$ is more palatable sufficient condition.

COROLLARY 2. <u>If</u> $\sum_{n=2}^{\infty} (1 + (n-1)\sec\lambda)|a_n| \le 1$, <u>then</u> $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in Sp(\lambda)$.

Using a different method, Corollary 2 will also be shown to follow from Theorem 3.

3. ORDER OF STARLIKENESS.

Since $B_n(\lambda) > n$, we see from Theorem A that a function satisfying the conditions of Theorem 1 must be starlike. We can actually do better.

THEOREM 2. <u>If f satisfies the conditions of Theorem 1 then</u> $f \in S^*(\alpha_0)$ <u>for</u>

$\alpha_0 = (3 - \sqrt{1 + 8\cos^2\lambda})/2(1 + \cos\lambda)$. <u>The result is sharp, with extremal function</u>

$f_2(z) = z + z^2/B_2(\lambda)$.

PROOF. In view of Theorem A, we need only show that $B_n(\lambda) \ge (n-\alpha_0)/(1-\alpha_0)$ for every n. Since $B_2(\lambda) = (2-\alpha_0)/(1-\alpha_0)$, it suffices to prove that $(1-\alpha_0)B_n(\lambda)/(n-\alpha_0)$ is an increasing function of n. Setting

$G(x) = \dfrac{x-1 + \sqrt{(x-1)^2 + 4x\cos^2\lambda}}{x-\alpha_0}$, I will show that $G'(x) > 0$ for $x > 2$.

A differentiation of G leads to

$$(x-\alpha_0)^2 G'(x) = 1-\alpha_0 + \left[\frac{(1-\alpha_0-2\cos^2\lambda)x - (1-\alpha_0+2\alpha_0\cos^2\lambda)}{\sqrt{(x-1)^2 + 4x\cos^2\lambda}}\right]$$

= H(x), say.

Since $H'(x) = \dfrac{4\sin^2\lambda\cos^2\lambda(x-\alpha_0)}{((x-1)^2 + 4x\cos^2\lambda)^{3/2}} > 0$, it follows for $x > 2$ that

$$H(x) > H(2) = \cos\lambda\left(1 - \frac{4\cos\lambda - 1}{\sqrt{1+8\cos^2\lambda}}\right) > 0.$$

Therefore $G(x)$, and consequently $(1-\alpha_0)B_n(\lambda)/(n-\alpha_0)$, is an increasing function. This completes the proof.

We have actually shown more according to Theorem A.

COROLLARY. <u>If</u> f <u>satisfies the conditions of Theorem 1, then</u> $|(zf'/f) - 1| < 1-\alpha_0$, $z \in \Delta$.

Functions in $S^*(\alpha)$ need not be in $Sp(\lambda)$ for $\lambda \neq 0$. The function $f_\alpha(z) = z/(1-z)^{2(1-\alpha)} \in S^*(\alpha)$ since zf'_α/f_α maps Δ onto the half plane $\text{Re } w > \alpha$. However $f_\alpha \notin Sp(\lambda), \lambda \neq 0$. We next look at a subclass of $S^*(\alpha)$ whose functions are spiral-like.

THEOREM 3. <u>If</u> $f(z) = z + \ldots$ <u>is analytic with</u> $|(zf'/f) - 1| < 1-\alpha$ <u>for</u> $z \in \Delta$, then $f \in Sp(\lambda)$ <u>for</u> $|\lambda| < \cos^{-1}(1-\alpha)$. <u>The result is sharp, with extremal function</u> $f(z) = ze^{(1-\alpha)z}$.

PROOF. We may write $(zf'/f)-1 = (1-\alpha)\omega(z)$, where $|\omega(z)| < 1$ for $z \in \Delta$. Thus $\text{Re}\{e^{i\lambda}zf'/f\} = \cos\lambda + (1-\alpha)\ \text{Re}\{e^{i\lambda}\omega(z)\} > \cos\lambda - (1-\alpha)|e^{i\lambda}\omega(z)| > \cos\lambda - (1-\alpha) > 0$ for $|\lambda| < \cos^{-1}(1-\alpha)$, and the proof is complete.

COROLLARY. <u>If</u> $|(zf'/f) - 1| < \cos\lambda$, <u>then</u> $f \in Sp(\lambda)$.

PROOF. Set $\alpha = 1 - \cos\lambda$ in Theorem 3.

Finally an application of Theorem A, with $\alpha = 1 - \cos\lambda$, to Theorem 3 provides us with an alternate proof to Corollary 2 of Theorem 1.

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