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# A DECOMPOSITION INTO HOMEOMORPHIC HANDLEBODIES WITH NATURALLY EQUIVALENT INVOLUTIONS

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ABSTRACT. Downing [6] extended the well-known result that any closed 3-manifold X contains a handlebody H such that cl(X-H) is homeomorphic to H in the case where X is a compact 3-manifold with nonvoid boundary. We show that if X is a compact 3-manifold with involution h having 2-dimensional fixed point set, then X contains an h-invariant handlebody H such that the involutions induced on H and cl(X-H) are naturally equivalent.

KEY WORDS AND PHARSES. 3-manifold, involution, handlebody, equivalent involutions.
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## 1. INTRODUCTION.

In this paper all spaces and maps are piecewise linear (PL) in the sense of Rourke and Sanderson [1]. Thus all spaces can be triangulated and all maps and subspaces are PL. This is no restriction as every 3-manifold has a triangulation by Bing [2]. Notation follows Nelson [3], [4], and Rourke and Sanderson [1].

Let X be a closed 3-manifold with triangulation T. If S is the 1-skeleton of T and $H_1$ is a regular neighborhood of S, then it is a classical result, see Seifert and Threlfall [5,p.219], that $H_1$ and $H_2$ = cl(X-H) are homeomorphic handlebodies. Downing [6] extended this result, showing that every 3-manifold with nonvoid boundary also contains a handlebody which is homeomorphic to its complement. It is the purpose of this paper to show that these results can be further extended so that certain symmetries are respected. These symmetries are those realized as involutions of X with orientable 2-dimensional fixed point sets.

Recall that an involution of X is a homeomorphism on X of period two. Involutions $h_i:X_i \to X_i$, i = 1, 2, are equivalent, denoted $h_1 \sim h_2$ , if there exists a homeomorphism $f:X_1 \to X_2$ such that $h_1 = f^{-1} \circ h_2 \circ f$. The fixed point set of involution $h:X \to X$ is $F(h) = \{x \in X | h(x) = x\}$. We shall prove the following:

THEOREM 1. Let X be an orientable 3-manifold and $h:X \to X$ an involution with 2-dimensional, orientable fixed point set F. Then there exist complementary, h-invariant handlebodies $H_1$ and $H_2$ in X such that $h|H_1 \sim h|H_2$.

THEOREM 2. Let $f:H_1 \to H_2$ be the homeomorphism such that $h|H_1 = f^{-1} \circ (h|H_2) \circ f$ thereby establishing the equivalence $h|H_1 \sim h|H_2$ of Theorem 1. Then $f|\partial H_1$ may be assumed to be the identity on $\partial H_1 = H_1 \cap H_2 = \partial H_2$.

By Nelson [4; Thm. C] the involutions $h|H_1$ and $h|H_2$ are equivalent if and only if (i) their fixed point sets are homeomorphic and (ii) either both fixed point sets separate or both fail to separate. In Nelson [3; lemma 2.1] it was shown that a 2-dimensional fixed point set of an involution on an orientable handlebody separates the handlebody if and only if the 1-dimensional fixed point set of the involution induced on the boundary separates the boundary. Since $H_1$ and $H_2$ share a common boundary, we need only show that $H_1$ can be chosen so that $H_1 \cap F \approx H_2 \cap F$. Hence, Theorem 1 follows immediately from the following lemma:

LEMMA 1. Let X be a compact 3-manifold and $h:X \to X$ an involution with 2-dimensional, orientable fixed point set F. Then there exist complementary, h-invariant handlebodies $H_1$ and $H_2$ in X such that $H_1 \cap F \approx H_2 \cap F$.

The 3-manifold X may be either orientable or nonorientable in this lemma. X must be assumed to be orientable in Theorem 1 in order to invoke Nelson [4; Thm. C]. Section 2 contains the proof of this lemma in the case when X is a closed manifold. The proof is modified in section 3 to cover the case when X has nonvoid boundary. The proof of Theorem 2 is provided in section 4. Both sections 3 and 4 are modifications of proofs which appear elsewhere. These sections refer heavily to the proofs which they modify.

2. X IS CLOSED.

Our approach is to pick an initial handlebody $H_1$ that will be repeatedly modified until $H_1 \cap F_i \approx H_2 \cap F_i$ for each component $F_i$ of F. This identity is established by showing $H_1 \cap F_i^+ \approx H_2 \cap F_i^-$ and $H_1 \cap F_i^- \approx H_2 \cap F_i^+$, where $F_i^+$ and $F_i^-$ are "homeomorphic halves" of $F_i$.

The Initial $H_1$ : We note first that any component $F_i$ of F is a closed, orientable surface. Any such surface admits an involution $g_i$ with a separating fixed point set consisting of at most two simple closed curves. This separates $F_i$ into $F_i^-$ and $F_i^+$ where $g_i(F_i^-) = F_i^+$.

Let $X_h^*$ be the orbit space of h. Since F is 2-dimensional, $X_h^*$ is a bordered 3-manifold with $\partial X_h^* \approx F$. Triangulate $\partial X_h^*$ so that the triangulation of component $F_i$ is $g_i$-invariant. Since $\partial X_h^*$ is collared, Rouke and Sanderson [1; Cor. 2.26], the triangulation of $\partial X_h^*$ can be extended to a triangulation T* of $X_h^*$. Let T be the lift of T* to X. Then T is an h-invariant triangulation of pair (X,F) such that $F_i^+$ is triangulated identically to $F_i^-$ for each i.

Denote by $T^{(n)}$ the nth derived subdivision of T and by $S^{(n)}$ the collection of all edges and vertices of $T^{(n)}$ lying on the 1-skeleton S of T. Let $H_1 = N(S^{(2)}, T^{(2)})$ and $H_2 = (X-H_1)$. $H_1$ and $H_2$ are homeomorphic orientable handlebodies. Since T triangulates F, $H_1 \cap F_i$ is the connected, orientable surface $F_i$ minus k disks, where k is the number of 2-simplices in the triangulation of $F_i$. $H_2 \cap F_i$ consists of k disks.

Modification Procedures: Two procedures are used to repeatedly modify T near each $F_i$. The result of modification by either procedure is that the modified handlebody $\tilde{H}_1$ is isomorphic to $\tilde{H}_1$ plus a 3-handle of index one and the new complementary handlebody $\tilde{H}_2$ is isomorphic to $H_2$ plus a 3-handle of index one. Hence, after each modification we are again left with homeomorphic, complementary handlebodies.

The 3-simplices of T are in h-invariant pairs. We say such a pair is near $F_i$ if the two 3-simplices of the pair intersect in $F_i$. The procedures below correspond to the cases where 3-simplices intersect in an edge or 2-face in $F_i$. We will not need to employ modifications exploiting simplices intersecting $F_i$ in a vertex.

Procedure α: The net result of this procedure is to remove one edge from $T \cap F_i$. Intuitively, $\tilde{H}_1 \cap F_i$ contains "less" of $F_i$ than $H_1 \cap F_i$.

Let e be any edge of T lying in $F_i$. Then e is the common edge of two 3-simplices, $s_1$ and $s_2$, such that $h(s_1) = s_2$ and $s_1 \cap s_2 = \{e\}$. Let $e_1$ and $e_2$ be the edges of $T^{(1)}$ connecting the vertices of e with the barycenter of $s_1$. Modify $S^{(2)}$ to get $\hat{S}^{(2)}$ by deleting from $S^{(2)}$ all vertices and edges of $T^{(2)}$ lying on e and adding to $S^{(2)}$ all vertices and edges of $T^{(2)}$ lying on $(e_1 \cup e_2) \cup h(e_1 \cup e_2)$. We replace $H_1$ by $\tilde{H}_1 = N(\hat{S}^{(2)}, T^{(2)})$. $\tilde{H}_1$ is homeomorphic to $H_1$ plus a 3-handle of index one. $\tilde{H}_2 = cl(X-\tilde{H}_1)$ is homeomorphic to $H_2$ plus a 3-handle of index one, the cocore of which lies in a disk bounded by $(e_1 \cup e_2) \cup h(e_1 \cup e_2)$.

Procedure β: The net result of this procedure is to add an h-invariant edge to T which intersects $F_1$ in a point. As a result, $\tilde{H}_1 \cap F_i$ contains a disk component not in $H_1 \cap F_i$.

Let $a_1a_2a_3$ be any 2-simplex of T in F. Then $a_1a_2a_3$ is the common face of 3-simplices $a_1a_2a_3a_4$ and $a_1a_2a_3a_5$ such that $h(a_4) = a_5$. The union of these simplices is a double tetrahedron we shall retriangulate. The new triangulation consists of the simplices $a_1a_2a_4a_5$, $a_1a_3a_4a_5$, $a_2a_3a_4a_5$ and their faces. It is easily checked that this new triangulation is h-invariant. Let $\tilde{T}$ be the triangulation of X obtained by replacing the former simplices of the double tetrahedron by the new ones. Substitute for $H_1$, $\tilde{H}_1 = |N(\tilde{S}^{(2)}, \tilde{T}^{(2)})|$. $\tilde{H}_1$ is homeomorphic to $H_1$ plus a 3-handle whose core is the edge $a_4a_5$. $\tilde{H}_2 = cl(X-H_2)$ is also homeomorphic to $H_1$ plus a 3-handle of index one.

Employing the Modifications: The α and β modifications are repeated independently near each $F_i$. Let f be the number of 2-simplices and p the number of vertices in the triangulation of $F_i^+$ (and also of $F_i^-$). We assume that the original T was chosen so that $p < 2f$ for each fixed point component $F_i$.

CASE 1. $p > f$. First apply $\beta$ p-f times where $a_1a_2a_3 \in F_i^-$. In order to carry this out we must have that $p-f \leq f$. But this is equivalent to the statement that $\chi(F_i^-) \leq 3f - e$ which is immediate when $\chi(F_i^-) = \frac{1}{2}\chi(F_i) \leq 0$ and easily checked in the remaining case when $F_i \simeq S^2$. Next, apply procedure $\alpha$ to every edge in $F_i^-$. Then both $H_1 \cap F$ and $H_2 \cap F$ consist of p disk components and one component homeomorphic to $F^+$(or $F^-$) minus p disks.

CASE 2. $f > p$. First apply procedure $\beta$ f-p times to simplices in $F_i^+$. Then use procedure $\alpha$ to remove all edges of T from $F_i^+$. The result is that both $H_1 \cap F$ and $H_2 \cap F$ consist of f disk components and one component homeomorphic to $F^+$ (or $F^-$) less f disks.

## 3. X IS NONVOID.

In this section we make the adjustments necessary in the above proof for the case $\partial X \neq \phi$. Notice that only the initial choice of $H_1$ was dependent upon X having empty boundary. The modification procedures do not depend upon assumptions about $\partial X$.

First, it was noted that $F_i$ is a closed, orientable surface. If $\partial X \neq \phi$, this is no longer necessarily true. However, $F_i$ will be a closed orientable surface less a collection of disks. Such a surface also admits an involution $g_i$ with a separating fixed point set consisting of simple closed curves and arcs.

No longer is it true that a regular neighborhood of the 1-skeleton of triangulation of X will have a homeomorphic complement in X. Rather, we must show that Downing's construction of the handlebody $H_1$ can be carried out equivariantly.

Let $B_i$, $i = 1, \ldots, n$ be the boundary components of X and let $X' = X \cup_k (\cup M_i)$, where $M_i$ is the handlebody with $\partial M \simeq B_i$ and $k|B_i$ mapping $B_i$ homeomorphically to $\partial M_i$. X' is a closed 3-manifold and accordingly has a classical decomposition into homeomorphic handlebodies $H_1'$ and $H_2' = cl(X' - H_1')$. By Downing [6; Lemma 1], each $M_i$ is isotoped by G in X' onto $N_i$ where $N_{ij} = N_i \cap H_j'$, $j = 1,2$, are specially positioned in $H_1$ and $H_2$. ($M_i$ and $N_i$ are, respectively, just regular neighborhoods of the 1-dimensional sets denoted Y and $Y^1$ in Downing's proof.) It is then shown that $X \simeq X' - \bigcup_i M_i$ and that $H_1' = H_1 - \bigcup_i N_{i1}$ and $H_2' = H_2 - \bigcup_i N_{i2}$ are homeomorphic handlebodies with $H_2 = cl((X' - \bigcup_i N_i) - H_1)$.

The isotopy G of X' moving each $M_i$ to $N_i$ yields a homeomorphism $f: X' \to X'$ such that $f(N_i) = M_i$ for all i. $f(H_1)$ and $f(H_2)$ are homeomorphic handlebodies such that $f(H_2) = cl(x - f(H_1))$. We do not yet know, however, that $f(H_1)$ and $f(H_2)$ are h-invariant. $H_1'$ and $H_2'$ may, however, be assumed h-invariant as in section 2. But if (i) h can be extended to an involution $h': X' \to X'$ and (ii) if the isotopy G can be constructed to commute with h', then $f(H_1)$ and $f(H_2)$ will be h-invariant as needed.

To extend h to h' we need only know that any involution (free or with 1-dimensional fixed point set) on $\partial M_i$ is equivalent by $k_i = k|B_i$ to the restriction to $B_i$ of an involution on $M_i$. Then attaching $M_i$ to X by $k_i$ allows h to be extended into $M_i$. Let $M_i = A \times I$ where A is a 2-sphere minus disks. If h is a free involution of $\partial M_i$, then $h|M_i$ is equivalent to $\ell$ given $\ell(m,t) = (\alpha(m),t)$ where $\alpha$ is the free involution on A derived from the antipodal map on $S^2$ by removing invariant pairs of disks.

If $h|\partial M_i$ has a 1-dimensional fixed point set, it will consist only of simple closed curves. In this case $h|M_i$ is equivalent to $\ell$ where $\alpha$ is an involution of A with arcs as fixed point components. If the fixed point set separates, than $\alpha$ may be derived from the reflection in a great circle of $S^2$ by removing invariant disks from the fixed point circle and invariant pairs of disks disjoint from the circle of fixed points.

Suppose $h|M_i$ has a nonseparating fixed point set consisting of k simple closed curves. Let $M_i = A \times I$ where A is $S^1 \times S^1$ minus disks. The involution $m: S^1 \times S^1 \to S^1 \times S^1$ given by $\gamma(s_1, s_2) = (s_2, s_1)$ has one nonseparating fixed point loop. Again, $h|M_i \sim \alpha$ where $\alpha$ is derived from $\gamma$ by removing k invariant disks from the nonseparating fixed point and an appropriate number of invariant pairs of disks disjoint from the fixed point set.

Let $A \times [-1,1]$ be embedded in X' by j such that $h'|j(A \times [-1,1])$ preserves the locally defined product structure. Using an invariant collar C on $B_i$ in X, j can be extended to an embedding, also called j, of $A \times [-1,3]$ into X', such that $j(A \times \{1\}) = B_i$ and so that $h'|j(A \times [-1,3])$ is fiber preserving. By Kim and Tollefson [7; Thm. B], $h'|C \sim \ell$ where $\ell: B_i \times [-1,3] \to B_i \times [-1,3]$ is given by $\ell(b,t) = ((h'|B_i)(b),t)$. Since $j(A \times \{1\})$ is an invariant subspace of $B_i$, $j(A \times [1,3])$ is an invariant subspace of C with $h'|j(A \times [1,3])$ fiber preserving. Replacing $T \times [-1,3]$ in the proof of Downing [6; lemma 1] by $A \times [-1,3]$, the isotopy $G_t$, with support in $j(A \times [-1,3])$, taking $M_i$ to $N_i$ may be assumed to commute with h'. That is, we may assume that $G_t$ preserves the fiber structure on $j(A \times [-1,3])$.

In the case where the boundary component $B_i$ is not h-invariant, such care in choosing the isotopy $G_t$ is unnecessary. Any isotopy $G_t$, with support in a small neighborhood of $M_i$, will suffice as long as $h'G_t$ is applied near $M_j = hM_i$.

## 4. PROOF OF THEOREM 2.

In this section we show that the equivalence $h|H_1 \sim h|H_2$ is in some sense natural. That is, we show that the homeomorphism $f: H_1 \to H_2$ establishing this equivalence may be assumed to be the identity on $\partial H_1 = H_1 \cap H_2 = \partial H_2$. This we do by modifying the proofs of Nelson [4, Theorem B] and Nelson [3, Theorem 4.1].

The proof of the equivalence in the above referenced papers is by induction on the common genus of the handlebodies $H_1$ and $H_2$. Nelson [4; Lemma 2.1] provides $(h|H_i)$-invariant, nonseparating, properly embedded disks $D_i \subset H_i$ such that cutting along the $D_i$ lowers the genera of the fixed point sets $F(h|H_i)$, i = 1, 2. Let $H_i' = H_i - N(D_i)$, i = 1, 2, be the handlebodies resulting from cutting along $D_i$. By the induction hypothesis, $h|H_1' \sim h|H_2'$ and also $h|N(D_1) \sim h|N(D_2)$. Let these equivalences be established by $f_1: H_1' \to H_2'$ and $f_2: N(D_1) \to N(D_2)$. By the homogeneity of the manifolds $(H_1')^*$ and $N(D_1)^*$, $i = 1, 2$, one may assume that $f_1(H_1' \cap N(D_1)) = f_2(H_2' \cap N(D_2))$. The equivalence $h|H_1 \sim h|H_2$ is established by $f: H_1 \to H_2$ which is reconstructed from the homeomorphism $f_1$ and $f_2$.

In order to modify the above argument so that $f|\partial H_1$ is the identity on $H_1 \cap H_2$ we need only to be able to choose $D_1 \subset H_1$ and $D_2 \subset H_2$ so that $\partial D_1 = \partial D_2$. The remainder of this proof consists in showing that $D_1$ and $D_2$ can be so chosen.

Consider the disk $D_i$ properly embedded in $H_i$ and cutting $F(h|H_i)$ as provided by Nelson [4;Theorem B]. Let p denote the projection onto the orbit space $H_i^*$. Then $p(D_i)$ is a disk properly embedded in $H_i^*$. $\partial p(D_i) = \alpha_i \cup \beta_i$, the union of two arcs, where $\alpha_i \subset p(F(h\ H_i))$ and $\beta_i \subset p(\partial H_i)$. $D_2$ can be isotoped onto $D_1$ relative to F if and only if $\alpha_2$ can be isotoped onto $\alpha_1$ relative to $p(F(h|H_2))$ and $\beta_2$ can be isotoped onto $\beta_1$ relative to $p(\partial H_2)$. We can find such $D_1$ and $D_2$ by the following procedure.

Fix $D_1$ and regard $\partial D_1$ as a fixed loop on $\partial H_2$. Employ Nelson [4; Theorem B] to find $D_2 \subset H_2$. If $\alpha_2$ and $\beta_2$ can be isotoped relative to $p(F(h|H_2))$ onto $\alpha_1$ and $\beta_1$, respectively, then we are done. If not, cut $H_2$ along $D_2$ to get a handlebody $H_2' = cl(H_2 - N(D_2))$ of lower genus. $\partial D_1$ may still be regarded as a fixed loop on $\partial H_2'$. Note that with this cut the bordered surfaces $p(F \cap H_2')$ and $p(\partial H_2')$ which make up $\partial(H_2')^*$ have lower genera than $p(F \cap H_2)$ and $p(\partial H_2)$, respectively. Now apply Nelson [4, Theorem B] again to $H_2'$ taking additional care that $D_2'$ misses the "scars" of previous cuts (so that $D_2'$ may be regarded as properly embedded in $H_2$ ). This can be done because of the homogeneity of $\partial(H_2')^*$. Repetition of this procedure must eventually yeild the desired disks $D_1$ and $D_2$ since the procedure eventually terminates with both $p(\partial H_1')$ and $p(F\ \ H_2')$ being disks.

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