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ON THE δ-CONTINUOUS FIXED POINT PROPERTY

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ABSTRACT. In this paper, we define and investigate the δ-continuous retraction and the δ-continuous fixed point property. Theorem 1 of Connell [11] and Theorem 3.4 of Arya and Deb [2] are improved.

KEY WORDS ANDPHRASES. δ-continuous , θ-continuous , weakly-continuous, semi-regular, almost-regular, fixed point property.
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0. INTRODUCTION.

The notion of θ-continuous functions was first introduced by Fomin [1]. After that, this notion has been widely investigated in the literature. By utilizing θ-continuous functions, Arya and Deb [2] defined and investigated the θ-continuous retraction, the θ-continuous fixed point property and the θ-continuous homotopy. On the other hand, in [3] and [4] the present authors have independently introduced the notion of δ-continuous functions. The purpose of this paper is to apply δ-continuity to the retraction and the fixed point property. In Section 2, we study the retraction of a topological space by δ-continuous functions. Section 3 deals with the fixed point property in relation to δ-continuous functions. The main results of this paper are Theorems 3.2 and 3.3 which improve Theorem 1 of [11] and Theorem 3.4 of [2], respectively.

1. PRELIMINARIES.

Throughout the present paper, spaces will always mean topological spaces on which no separation axioms are assumed unless explicitly stated. We shall denote a topological space by (X, τ) or simply by X. Let (X, τ) be a space and A a subset of X.The closure of A and the interior of A are denoted by $\overline{A}^{\tau}$ and $\mathring{A}^{\tau}$ (or simply $\overline{A}$ and $\mathring{A}$), respectively. A subset A of X is said to be *regular open* (resp. *regular closed*) if $A=(\overline{A})^{\circ}$ (resp. $A = \overline{\mathring{A}}$). The family of regular open sets of X will be denoted by RO(X). A point x of X is said to be in the δ- *closure* [5] of A, denoted by $Cl_{\delta}(A)$, if $A \cap V \neq \emptyset$ for every $V \in RO(X)$ containing x. A subset A is said to be δ-*closed* [5] if $A = Cl_{\delta}(A)$. The complement of a δ-*closed* set is said to be δ - *open*. The topology on X which has RO(X) as a basis is called the *semi-regularization* of τ and is denoted by τ^{*}. It is obvious that every element of τ^{*} is a δ - *open* set of (X, τ). A space (X, τ) is said to be *semi-regular* if $\tau = \tau^{*}$. A space (X, τ) is said to be *almost-regular* [6] if for each regular closed set F and each $x \in X - F$, there exist open sets U and V such that $x \in U$, $F \subset V$ and $U \cap V = \emptyset$

DEFINITION 1.1. A function $f: (X, \tau) \to (Y, \sigma)$ is said to be δ-*continuous* [3, 4] (resp. *almost-continuous* [7], θ-*continuous* [1] and *weakly continuous* [8]) if for each $x \in X$ and each open neighborhood V of $f(x)$, there exists an open neighborhood U of x such that $f(\mathring{\overline{U}}) \subset \mathring{\overline{V}}$ (resp. $f(U) \subset \mathring{\overline{V}}$, $f(\overline{U}) \subset \overline{V}$ and $f(U) \subset \overline{V}$).

REMARK 1.1. It is shown in [2, 3, 9] that the following implications hold: δ-*continuous* $\Rightarrow$ *almost-continuity* $\Rightarrow$ θ-*continuous* $\Rightarrow$ *weak-continuity*, where none of these implications is reversible.

2. δ - CONTINUOUS RETRACTIONS.

Arya and Deb [2] defined a subset A of a space X to be a θ-continuous retract of X if there exists a θ-continuous function $f: X \to A$ such that f/A is the identity on A. We shall similarly define a δ-continuous retract.

DEFINITION 2.1. A subset A of space X is called a δ-*continuous retract* of X if there exists a δ-continuous function $f: X \to A$ such that f is the identity on A, that is, $f(x) = x$ for every $x \in A$. And such a function f is called a δ-*continuous retraction*.

REMARK 2.1. It is obvious that every δ-continuous retract is a θ-continuous retract. However, every δ-continuous retract is not necessarily a continuous retract as the following example shows.

EXAMPLE 2.1. Let $X = \{a, b, c, d\}$ and $t = \{\Phi, X, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$. Let $A = \{a, b, c\}$ and $f: (X, \tau) \to (A, \tau / A)$ be a function defined as follows : $f(a) = a$, $f(b) = b$, $f(c) = c$ and $f(d) = d$. Then A is a δ-continuous retract of X but it is not a continuous retract of X since $f^{-1}(\{a\}) \notin \tau$ for $\{a\} \in \tau / A$.

REMARK 2.2. In Example 3.1 of [2], Arya and Deb showed that every θ-continuous retracts is not necessarily a continuous retract. However, this example is false. The θ-continuous function $f: X \to A$ in [2, Example 3.1] is necessarily continuous since the subspace A is discrete and regular. Since every δ-continuous function is θ-continuous, Example 2.1 also shows that every θ-continuous retract is not a continuous retract.

We shall investigate relationships between δ-continuous retract and continuous retract.

PROPOSITION 2.1. If X is a semi-regular space and A is a continuous retract of X, then A is a δ-continuous retract of X.

PROOF. This follows from the fact that a continuous function from a semi-regular space is δ-continuous [3, Prop. 1.5].

LEMMA 2.1. If A is either open or dense in a space X and $V \in RO(X)$, then $V \cap A$ is regular open in the subspace A.

PROOF. If A is dense in X, then this follows from [10, p. 175, B)]. Next, suppose that A is open in X and $V \in RO(X)$. Then, we have

$$\mathring{\overline{V \cap A}}^{(A)} = (\overline{V \cap A} \cap A)^{\circ (A)} = (\overline{V \cap A} \cap A)^{\circ} = \mathring{\overline{V \cap A}} \cap A.$$

Moreover, we have $\overline{\mathring{(V\cap A)}} \cap A \supseteq (V \cap \overline{A})^{\circ} \cap A = V \cap A$. On the other hand, $\overline{\mathring{(V\cap A)}} \cap A \subset (\overline{V} \cap \overline{A})^{\circ} \cap A = \mathring{\overline{V}} \cap A = V \cap A$.
Therefore, we obtain $\overline{\mathring{(V\cap A)}}^{(A)} = V \cap A$ and hence $V \cap A$ is regular open in A.

PROPOSITION 2.2. Let X be a semi-regular space and A either open or dense in X. Then A is a continuous retract of X if and only if A is a δ-continuous retract of X.

PROOF. From Lemma 2.1, for A either open dense and X semiregular, $\tau = \tau^*$ and $(\tau / A) = (\tau^* / A) \subseteq (\tau / A)^* \subseteq (\tau / A)$.
Therefore, A is semiregular so that $f : X \to A$ is δ-continuous if and only if it is continuous.

PROPOSITION 2.3. Let X be a space and A a semi-regular (resp. almost-regular) subspace of X. If A is a δ-continuous (resp. continuous) retract of X, then it is a continuous (resp. δ-continuous) retract of X.

PROOF. Let $f : X \to A$ be a δ-continuous retraction and A be semi-regular. Every δ-continuous function into a semi-regular space is continuous [3, Prop. 1.4]. Therefore, A is a continuous retract of X. Every continuous function into an almost regular space is δ-continuous [3, Prop. 1.8]. Therefore, the second part follows.

THEOREM 2.1. If A is a δ-continuous retract of X and B is a δ-continuous retract of A, then B is a δ-continuous retract of X.

PROOF. Let $f : X \to A$ and $g : A \to B$ be δ-continuous retractions. The composite function $g \circ f : X \to B$ is δ-continuous [3, Prop. 3.2]. Moreover, we have $(g \circ f)(x) = g(f(x)) = g(x) = x$ for every $x \in B \subset A$. Therefore, $g \circ f : X \to B$ is a δ-continuous retraction and hence B is a δ-continuous retract of X.

THEOREM 2.2. A subset A of a space X is a δ-continuous retract of X if and only if for every space Y, every δ-continuous function $f : A \to Y$ can be extended to a δ-continuous of X into Y.

PROOF. Necessity. Let $g : X \to A$ be a δ-continuous retraction. Let Y be any space and $f : A \to Y$ be any δ-continuous function. Then composite function $f \circ g : X \to Y$ is δ-continuous [3, Prop. 3.2]. Moreover, we have $(f \circ g)(x) = f(g(x)) = f(x)$ for every $x \in A$. Therefore, $f \circ g$ is an extension of f.

Sufficiency. Let $i_A : A \to A$ be the identity function on A. Then i_A is δ-continuous and hence by the hypothesis there exists a δ-continuous function $g : X \to A$ such that $g / A = i_A$. Therefore, A is a δ-continuous retract of X.

THEOREM 2.3. If A is a δ-continuous retract of a Hausdorff space X, then A is δ-closed in X.

PROOF. Let $f : X \to A$ be a δ-continuous retraction. Suppose that A is not δ-closed in X. There exists a point $x \in Cl_\delta(A) - A$. Since $x \notin A$, $f(x) \neq x$ and hence there exist open sets U and V such that $x \in U$, $f(x) \in V$ and $U \cap V = \Phi$; hence $\mathring{\overline{U}} \cap \mathring{\overline{V}} = \Phi$. Let W be any regular open set containing x. Then $\mathring{\overline{U}} \cap W$ is a regular open set containing x. Since $x \in Cl_\delta(A)$, $[\mathring{\overline{U}} \cap W] \cap A \neq \Phi$. Let $a \in [\mathring{\overline{U}} \cap W] \cap A$, then $f(a) = a \in \mathring{\overline{U}}$ and hence $f(a) \notin \mathring{\overline{V}}$. This shows that $f(W) \not\subset \mathring{\overline{V}}$ for any regular open set W containing x. This contradicts the fact that f is δ-continuous.

3. THE δ-CONTINUOUS FIXED POINT PROPERTY.

Arya and Deb [2] defined a space X to have the θ-continuous fixed point property if, for every θ-continuous function $f : X \to X$, there exists an $x \in X$ such that $f(x) = x$. We shall define the δ-continuous (resp. weakly continuous) fixed point property as follows :

DEFINITION 3.1. A space X is said to have the δ-continuous (resp. weakly continuous) fixed point property, briefly denoted by δ cFPP (resp. wcFPP), if for every δ-continuous (resp. weakly continuous) function $f : X \to X$, there exists an $x \in X$ such that $f(x) = x$.

REMARK 3.1. It is obvious that a space with the wcFPP has necessarily the θ-continuous fixed point property and a space with the θ-continuous fixed point property has both the δcFPP and the fixed point property.

We give an example that a space with the fixed point property ned not have the δcFPP.

EXAMPLE 3.1. Let $X = \{a, b, c\}$ and $\tau = \{\Phi, X, \{a\}, \{a, b\}, \{a, c\}\}$. Then the space (X, τ) has the fixed point property [2, Example 3.2]. Now, let $f:(X, \tau) \to (X, \tau)$ be a function defined by $f(a) = f(c)=b$ and $f(b)=c$. Then f is δ-continuous but does not a fixed point. Therefore, (X, τ) does not have the δcFPP.

REMARK 3.2. We need the following two spaces which we were unable to obtain:

(1) a space which has δcFPP but does not have the fixed point property.

(2) a space which has the θcFPP but does not have the wcFPP.

THEOREM 3.1. Let A be either open or dense in a space X. If X has the δcFPP and A is a δ-continuous retract of X, then A has the δcFPP.

PROOF. Let $f: A \to A$ be any δ-continuous function. Since A is a δ-continuous retract of X, by Theorem 2.2 f can be extended to a δ-continuous function $F : X \to A$. Let $j : A \to X$ be the inclusion. If V is a regular open set of X, then $j^{-1}(V) = A \cap V$ is regular open in the subspace A by Lemma 2.1.Therefore, $F^{-1}(j^{-1}(V)) = (joF)^{-1}(V)$ is δ-open in X and hence $joF : X \to X$ is δ-continuous. Since X has the δcFPP, $x = (joF)(x) = j(F(x)) = j(f(x)) = f(x)$ for some $x \in A \subset X$. This shows that A has the δcFPP. The following theorem is a slight modification of Theorem 1 of [11].

THEOREM 3.2. Let (X, τ) be an almost-regular space with the δcFPP. If σ is a topology for X stronger than τ and $\overline{G}^{(\tau)} = \overline{G}^{(\sigma)}$ for every $G \in \sigma$, then (X, σ) has the fixed point property.

PROOF. Suppose that $f : (X, \sigma) \to (X, \sigma)$ is any continuous function. Let $g : (X, \sigma) \to (X, \tau)$ and $h : (X, \tau) \to (X, \tau)$ be the functions defined by $g(x) = h(x) = f(x)$ for every $x \in X$. Let $i : (X, \tau) \to (X, \sigma)$ be the identity function. Then, since $\tau \subset \sigma$, i is an open bijection. Moreover since $f = i \circ g$ is continuous, g is continuous. Next,we shall show that h is δ-continuous . Let $x \in X$ and $h(x) \in V \in RO(X, \tau)$. Since (X, τ) is almost-regular, there exists $G \in \tau$ such that $h(x) \in G \subset \overline{G}^{(\tau)} \subset V$. Since g is continuous, $g^{-1}(G) \in \sigma$ and $h^{-1}(G) = f^{-1}(G) = g^{-1}(G)$. Therefore, $h^{-1}(G) = f^{-1}(G) \in \sigma$ and hence, utilizing continuity of f we obtain $x \in h^{-1}(G) \subset \overline{h^{-1}(G)}^{(\sigma)\,o(\sigma)} = \overline{h^{-1}(G)}^{(\tau)\,o(\tau)} \subset \overline{h^{-1}(G)}^{(\tau)} = \overline{h^{-1}(G)}^{(\sigma)} = \overline{f^{-1}(G)}^{(\sigma)} \subset f^{-1}(\overline{G}^{(\sigma)}) \subset f^{-1}(\overline{G}^{(\tau)}) \subset f^{-1}(V) = h^{-1}(V)$. Now, we set $U = \overline{h^{-1}(G)}^{(\tau)\,o(\tau)}$,then we have $x \in U \in RO(X, \tau)$ and $h(U) \subset V$. This shows that h is δ-continuous. Since (X, τ) has the δcFPP, there exists $x \in X$ such that $x = h(x) = f(x)$. This shows that (X, σ) has the fixed point property.

COROLLARY 3.1 (Connell [11]). Suppose (X, τ) is a regular space with the fixed point property. If σ is a topology for X, $\tau \subset \sigma$ and $\overline{G}^{(\sigma)} = \overline{G}^{(\tau)}$ for each $G \in \sigma$, then (X, σ) has the fixed point property.

PROOF. It is shown in [3, Corollary 1.8] that if Y is regular, then $f : X \to Y$ is δ-continuous if and only if is continuous. Since every regular space is almost regular, this is an immediate consequence of theorem 3.2. We shall give a lemma which will be used in the proof of the final theorem.

LEMMA 3.1. Let $f : X \to Y$ and $g : Y \to Z$ be functions:

(1) f is weakly continuous if and only if $\overline{f^{-1}(V)} \subset f^{-1}(\overline{V})$ for each open set V of Y.

(2) If the composite $g \circ f : X \to Z$ is weakly continuous and $g : Y \to Z$ is an open bijection, then f is weakly continuous.

PROOF. Statement (1) is Theorem 7 of [12]. We shall show Statement (2) by utilizing Statement (1). Let V be any open set of Y. Since g is open, $g(V)$ is open in Z and $\overline{(g \circ f)^{-1}(g(V))} \subset (g \circ f)^{-1}(\overline{g(V)})$. Since g is bijective, $(g \circ f)^{-1}(g(V)) = f^{-1}(V)$. Moreover, since g is open, $(g \circ f)^{-1}(\overline{g(V)}) = f^{-1}(g^{-1}(\overline{g(V)})) \subset f^{-1}(\overline{g^{1}(g(V))}) = f^{-1}(\overline{V})$.Consequently, we obtain $\overline{f^{-1}(V)} \subset f^{-1}(\overline{V})$ and hence f is weakly continuous.

The following theorem is an improvement of [2, Theorem 3.4] and [11, Theorem 1].

THEOREM 3.3. Let (X, τ) be a regular space with the fixed point property. If σ is a topology for X stronger than τ and $\overline{G}^{(\sigma)} = \overline{G}^{(\tau)}$ for every $G \in \sigma$, then (X, σ) has the wcFPP.

PROOF. Let $f : (X, \sigma) \to (X, \sigma)$ be any weakly continuous function. Let $g : (X, \sigma) \to (X, \tau)$,

$h:(X,\tau)\to(X,\tau)$ and $i:(X,\tau)\to(X,\sigma)$ be the same functions as in Proof of Theorem 3.2. Since $f=i\circ g$ is weakly continuous and i is an open bijection, g is weakly continuous by Lemma 3.1. Since (X,τ) is regular, g is continuous [8]. Next, we shall show that h is continuous. Let $x\in X$ and V be an open set of (X,τ) containing $h(x)$. Since (X,τ) is regular, there exists $G\in\tau$ such that $h(x)\in G\subset\overline{G}^{(\tau)}\subset V$. Since g is continuous, $g^{-1}(G)\in\sigma$ and $h^{-1}(G)=f^{-1}(G)=g^{-1}(G)$. Therefore, we have $h^{-1}(G)=f^{-1}(G)\in\sigma$. Since f is weakly continuous, by Lemma 3.1 $\overline{f^{-1}(G)}^{(\sigma)}\subset f^{-1}(\overline{G}^{(\sigma)})$. It follows from the same argument as in Proof of Theorem 3.2 that h is continuous. Since (X,τ) has the fixed point property, there exists a point $x\in X$ such that $x=h(x)=f(x)$. This shows that f has the fixed point property.

COROLLARY 3.2 (Arya and Deb [2]). If (X,τ) is a regular space with the fixed point property and if σ is a topology for X stronger than τ such that $\overline{G}^{(\sigma)}=\overline{G}^{(\tau)}$ for each $G\in\sigma$, then (X,σ) has the θ-continuous fixed point property.

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