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UNIFORM TOEPLITZ MATRICES

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ABSTRACT. We characterize all infinite matrices of bounded linear operators on a Banach space which preserve the limits of uniformly convergent sequences defined on an infinite set. Also, we give a Tauberian theorem for uniform summability by the Kuttner-Maddox matrix.

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1. INTRODUCTION.

By T we denote any infinite set of objects and we consider functions $f_k : T \to X$ for $k = 1, 2, \dots$, where $(X, ||.||)$ is a Banach space.

The notation $f_k \Rightarrow f$ will be used to signify that $f_k \to f$ as $k \to \infty$, uniformly on T, which is to say that there exists $f : T \to X$ such that for all $\varepsilon > 0$ there exists $k_o = k_o(\varepsilon) > 0$ with

$$||f_k(t) - f(t)|| < \varepsilon, \text{ for all } k > k_o \text{ and all } t \in T.$$

Now suppose that for $n,k = 1,2,\dots$ each $A_{nk} \in B(X)$, i.e. each A_{nk} is a bounded linear operator on X. Then we shall say that $A = (A_{nk})$ is a uniform Toeplitz matrix of operators if and only if:

$$\sum_{k=1}^{\infty} A_{nk} f_k(t) \text{ converges in the norm of } X$$

for each $n \in N = \{1,2,3,4,\dots\}$ and each $t \in T$ and

$$\sum_{k=1}^{\infty} A_{nk} f_k \Rightarrow f$$

whenever $f_k \Rightarrow f$.

Following Robinson [1] and Lorentz and Macphail [2], if (B_k) is a sequence in B(X) we denote the group norm of (B_k) by

$$||(B_k)|| = \sup ||\sum_{k=1}^{p} B_k x_k||$$

where the supremum is over all $p \in N$ and all x_k in the closed unit sphere of X.

By C we shall denote the (C,1) matrix of arithmetic means, given by

$$\begin{matrix} 1 & 0 & 0 & 0 & 0 & . & . & . \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & . & . & . \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{matrix}$$

By D we denote the Kuttner-Maddox matrix, used extensively in the theory of strong summability [3, 4, 5]:

$$\begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \end{matrix}$$

In work on strong summability it is often advantageous to use the fact that, for non-negative (p_k) the summability methods C and D are equivalent, in the sense that $p_k \to 0(C)$ if and only if $p_k \to 0(D)$.

In connection with Tauberian theorems we now introduce the idea of uniform strong slow oscillation.

Let $s_k : T \to X$ for each $k \in N$. Then we say that (s_k) has uniform strong slow oscillation if and only if $s_n - s_k \Rightarrow 0$ whenever $k \to \infty$ and $n > k$ with $n/k = O(1)$.

In what follows we shall regard s_k as the k-th partial sum of a given series of functions $\Sigma a_k = a_1 + a_2 + \ldots$, each $a_k : T \to X$.

2. UNIFORM TOEPLITZ MATRICES.

The following theorem characterizes the uniform Toeplitz matrices of operators which were defined in Section 1.

THEOREM 1. $A = (A_{nk})$ is a uniform Toeplitz matrix if and only if

$$\sup_n ||(A_{n1}, A_{n2}, \ldots)|| < \infty, \tag{2.1}$$

A is column-finite, (2.2)

$$\text{for each } n \in N,\ A_n := \sum_{k=1}^{\infty} A_{nk} \text{ converges,} \tag{2.3}$$

$$A_n = I, \text{ ultimately in } n. \tag{2.4}$$

PROOF. We remark that in (2.3) the convergence is in the strong operator topology, and in (2.4), I is the identity operator on X.

For the sufficiency, let H denote the value of the supremum in (2.1), let

$n \in N$ and $t \in T$. Then, for any $\varepsilon > 0$ there exists k_o such that $||f_k(t) - f(t)|| < \varepsilon$ for all $k > k_o$.

Now for each $p \in N$,

$$\sum_{k=1}^{p} A_{nk} f_k(t) = \sum_{k=1}^{p} A_{nk}(f_k(t) - f(t)) + \sum_{k=1}^{p} A_{nk} f(t),$$

where we assume that $f_k \Rightarrow f$. By (2.3), as $p \to \infty$, we have

$$\sum_{k=1}^{p} A_{nk} f(t) \to A_n f(t).$$

Also, if $s \geq r > k_o$,

$$|| \sum_{k=r}^{s} A_{nk}(f_k(t) - f(t))|| \leq H\varepsilon,$$

whence $\sum_{k=1}^{\infty} A_{nk} f_k(t)$ converges.

By (2.4) there exists $m \in N$ such that $A_n = I$ for all $n > m$, and by (2.2) there exists $n_o(\varepsilon) \in N$ such that $A_{nk} = 0$ for $1 \leq k \leq k_o$ and for $n > n_o(\varepsilon)$.

Taking $n > m + n_o$ we have

$$\sum_{k=1}^{\infty} A_{nk} f_k(t) = f(t) + \sum_{k=1+k_o}^{\infty} A_{nk}(f_k(t) - f(t)).$$

Since

$$|| \sum_{k=1+k_o}^{\infty} A_{nk}(f_k(t) - f(t))|| \leq \varepsilon||(A_{n1}, A_{n2}, \ldots)||,$$

it follows by (2.1) that $\Sigma A_{nk} f_k \Rightarrow f$, which proves the sufficiency.

Now consider the necessity. Take any convergent sequence (x_k) in X, with $x_k \to x$. Define $f_k(t) = x_k$ for all $k \in N$ and all $t \in T$, and define $f(t) = x$ for all $t \in T$. Then $f_k \Rightarrow f$ and so $\Sigma A_{nk} x_k$ converges for each n and tends to x, whence the usual Toeplitz theorem for operators, see Robinson [1] or Maddox [6], yields (2.1) and (2.3) of our present theorem.

Next, suppose that (2.4) is false. Then there exist natural numbers $n(1) < n(2) < \ldots$ with $A_{n(i)} \neq I$ for all $i \in N$. Hence there exist $x_i \in X$ with

$$||A_{n(i)} x_i - x_i|| > 0 \tag{2.5}$$

for all $i \in N$. Let us write $y(i)$ for the expression inside the norm bars in (2.5). Since T is an infinite set we may choose any countably infinite subset $\{t_1, t_2, t_3, \ldots\}$ of T. Then we define $f : T \to X$ by

$$f(t_i) = x_i/||y(i)|| \tag{2.6}$$

for all $i \in N$, and $f(t) = 0$ otherwise. If we define $f_k = f$ for all $k \in N$ then we certainly have $f_k \Rightarrow f$. But A is not a uniform Toeplitz matrix, since for $n = n(i)$ we have by (2.6),

$$\left\| \sum_{k=1}^{\infty} A_{nk} f(t_i) - f(t_i) \right\| = \|A_n x_i - x_i\| / \|y(i)\| = 1.$$

Hence, if A is a uniform Toeplitz matrix then (2.4) must hold, and a similar argument shows that (2.2) is necessary, which completes the proof of the theorem.

Since C, the (C,1) matrix, is not column-finite we immediately obtain:

COROLLARY 2. C is a Toeplitz matrix but not a uniform Toeplitz matrix.

However, since the elements of the Kuttner-Maddox matrix D are non-negative and its row sums all equal 1 it is clear that the conditions of Theorem 1 hold, whence D is a uniform Toeplitz matrix. Thus, whenever $f_k \Rightarrow f$ it follows that

$$2^{-r} \Sigma_r f_k \Rightarrow f, \tag{2.7}$$

where the sum in (2.7) is over $2^r \le k < 2^{r+1}$ for $r = 0,1,2,\ldots$. We also express (2.7) by writing $f_k \Rightarrow f(D)$.

The relation between C and D for uniform summability is given by:

THEOREM 3. $f_k \Rightarrow f(C)$ implies $f_k \Rightarrow f(D)$, but not conversely in general.

PROOF. Write

$$c(n) = n^{-1} \sum_{k=1}^{n} f_k(t) \quad \text{and} \quad d(r) = 2^{-r} \Sigma_r f_k(t).$$

Then we find that

$$d(r) = (2 - 2^{-r}) c(2^{r+1} - 1) - (1 - 2^{-r}) c(2^r - 1), \tag{2.8}$$

and it is clear that the right-hand side of (2.8) defines a uniform Toeplitz transformation between the c and d sequences.

For the last part of the theorem we may define real-valued functions on T by $f_k(t) = 2^r$ when $k = 2^r$ and $f_k(t) = -2^r$ when $k = 1 + 2^r$, and $f_k(t) = 0$ otherwise. Then $f_k \Rightarrow 0(D)$. Now suppose, if possible, that $f_k \Rightarrow f(C)$, which implies $f_k \Rightarrow f(D)$. Hence $f = 0$. But

$$c(2^r) - (1 - 2^{-r}) c(2^r - 1) = 1,$$

contrary to the fact that $c(n) \to 0$.

3. A UNIFORM TAUBERIAN THEOREM.

By the remark following Corollary 2 we know that $f_k \Rightarrow f$ implies $f_k \Rightarrow f(D)$, but the example of Theorem 3 shows that the converse is generally false. The next result shows that uniform strong slow oscillation is a Tauberian condition for uniform D summability.

THEOREM 4. If (s_k) has uniform strong slow oscillation and $s_k \Rightarrow f(D)$ then $s_k \Rightarrow f$.

PROOF. Without loss of generality we may suppose that $f = 0$.

Take $n \in N$ and determine r such that $2^r \leq n < 2^{r+1}$. If $\varepsilon > 0$ there exists r_o such that if $2^r \leq k < 2^{r+1}$ then

$$||s_k(t) - s_n(t)|| < \varepsilon$$

whenever $r > r_o$ and $t \in T$. Since

$$2^{-r}\Sigma_r s_k(t) = s_n(t) + 2^{-r}\Sigma_r(s_k(t) - s_n(t))$$

we see that $s_n \Rightarrow 0$.

Our final result shows that the natural conditions $ka_k \Rightarrow 0$ or $ka_k \Rightarrow 0(C,1)$ are both Tauberian conditions for uniform D summability, but that the restriction $ka_k \Rightarrow 0$ cannot be relaxed to the uniform boundedness of (ka_k).

THEOREM 5. (i) If $ka_k \Rightarrow 0$ or $ka_k \Rightarrow 0(C,1)$ and $s_k \Rightarrow f(D)$ then $s_k \Rightarrow f$.

(ii) There exists a divergent series Σa_k with (ka_k) uniformly bounded and $s_k \Rightarrow 0(D)$.

PROOF. (i) First note that $ka_k \Rightarrow 0$ does not generally imply $ka_k \Rightarrow 0(C,1)$ because (C,1) is not a uniform Toeplitz matrix by Corollary 2. We shall show that $ka_k \Rightarrow 0(C,1)$ is a Tauberian condition for D, the proof for $ka_k \Rightarrow 0$ being similar. In fact we shall show that $ka_k \Rightarrow 0(C,1)$ implies that (s_k) has uniform strong slow oscillation.

Let us write $a_k = a_k(t)$, $s_n = s_n(t)$ and

$$A_n = n^{-1}\sum_{k=1}^{n} ka_k,$$

with the assumption that $A_n \Rightarrow 0$. Then for $n > k \geq 1$, by partial summation,

$$s_n - s_k = A_n - \frac{k}{n}A_k + \sum_{\nu=k+1}^{n-1}(\nu A_\nu - kA_k)/\nu(\nu+1),$$

whence

$$||s_n - s_k|| \leq \max\{||A_\nu|| : k \leq \nu \leq n\}(1 + \frac{k}{n} + 2\sum_{\nu=k+1}^{n}\frac{1}{\nu+1}).$$

If $n/k = \mathcal{O}(1)$ then

$$1 + \frac{k}{n} + 2\sum_{\nu=k+1}^{n}\frac{1}{\nu+1} < 2 + 2\frac{n}{k} = \mathcal{O}(1),$$

and so $s_n - s_k \Rightarrow 0$, as required.

(ii) Define a numerical sequence (s_k) by $s_k = 0$ when $1 \leq k < 4$, and for $n \geq 2$

define $s_k = 0$ when $k = 2^n$ and when $k = 2^n + 2 \times 2^{n-2}$; $s_k = 1$ when $k = 2^n + 2^{n-2}$ and $s_k = -1$ when $k = 2^n + 3 \times 2^{n-2}$. Otherwise define s_k linearly, so that the graph of (s_k) is a triangular-shaped wave. Then (s_k) diverges and it is clear that $\Sigma_r s_k = 0$ for all $r \geq 0$. Also, it is easy to check that $k|a_k| \leq 8$ for all $k \geq 1$, whence our result follows on defining $s_k(t) = s_k$ for all $k \geq 1$ and all $t \in T$.

REFERENCES

1. ROBINSON, A. On functional transformations and summability. Proc. London Math. Soc. 52 (1950) 132-160.

2. LORENTZ, G.G. and MACPHAIL, M.S. Unbounded operators and a theorem of A Robinson. Trans. Royal Soc. of Canada. XLVI (1952) 33-37.

3. KUTTNER, B. and MADDOX, I.J. On strong convergence factors. Quarterly J. of Math. (Oxford). 16 (1965) 165-182.

4. MADDOX, I.J. On Kuttner's theorem. J. London Math. Soc. 43 (1968) 285-290.

5. MADDOX, I.J. Inclusions between FK spaces and Kuttner's theorem. Math. Proc. Camb. Phil. Soc. 101 (1987) 523-527.

6. MADDOX, I.J. Infinite matrices of operators, Springer-Verlag, 1980.