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OPERATIONAL CALCULUS AND DIFFERENTIAL EQUATIONS WITH INFINITELY SMOOTH COEFFICIENTS

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ABSTRACT. A subring M_F of the field of Mikusiński operators is constructed as a countable union space. Some topological properties of M_F are investigated. Then, the product of an infinitely differentiable function and an element of M_F is given and is used to investigate operational equations with infinitely smooth coefficients.

KEY WORDS AND PHRASES. Mikusiński operator, infinitely differentiable, operational equation.
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1. INTRODUCTION.

The theory of generalized functions has been used successfully in solving problems in classical analysis as well as in simplifying the theory of differential equations (Mikusiński [1], Zemanian [2,3]). In the field of Mikusiński operators, the inability to define a suitable product of a function and an operator, see Stankovic [4], has had a limiting effect. For example, the theory and applications of ordinary differential equations have been restricted to differential equations having constant coefficients. In this note we will confine ourselves to the field M of Mikusiński operators [1]. We will study a subring M_F of Mikusiński operators in which the product of an infinitely smooth function and an operator in M_F can be suitably defined. In Section 2 the construction of M_F and a convergence in M_F are studied. Then in Section 3, it is shown that an operational equation with infinitely smooth coefficients has only the classical solution. Some examples are then given to show that this is not the case if singularities are introduced.

2. THE CONSTRUCTION OF M_F AND CONVERGENCE.

The ring of continuous complex-valued functions on $[0,\infty)$, denoted by C, with addition and convolution $((f*g)(t) = \int_0^t f(t-u)g(u)du)$ has no zero divisors. The quotient field of C is denoted by M and is called the field of Mikusiński operators. A typical element x of M, called an operator, is written as $x = f/g$, where $f,g \in C$.

The integral operator ℓ is the function defined as 1 for $t > 0$ and 0 for $t < 0$. The integral operator has the property that, for all $f \in C$, $(\ell*f)(t) = \int_0^t f(u)du$. The inverse of ℓ, denoted by s, is called the differentiation operator.

ℓ^k will denote $\ell*\ell*\ell*....*\ell$ (k terms), while $\phi^{(k)}$ will denote the k^{th} derivative of ϕ.

For $k = 0,1,2,...$, let $M^k = \{x \in M: \ell^k*x \in C\}$. By endowing M^K with the topology induced by the countable family of seminorms

$\gamma_{k,m}(x) = \sup\{|(\ell^k*x)(t)|: 0 < t < m\}$ for $m = 1,2,...$ M^k is a Fréchet space. Clearly, $x_n \to x$ in M^k if and only if for each $m > 0$, $\ell^k*x_n \to \ell^k*x$ uniformly on $[0,m]$.

Let M_F be the countable union space $\bigcup_{k=0}^{\infty} M^k$. That is, x is an element of M_F if x is an element of M^k for some k. Also, a sequence $\{x_n\}$ in M_F is said to converge to an element x in M_F if for some some k, $x_n, x \in M^k$ $n=1,2,...$and $\{x_n\}$ converges to x in the topology of M^k. Since each M^k is complete, M_F is sequentially complete. For a more detailed discussion of countable union spaces see [2].

Even though M is considerably larger than M_F, M_F contains many of the important operators needed for applications. For, by identifying the locally integrable function f with the operator $\ell*f/\ell$, the collection of locally integrable functions can be identified with a subring of M_F. Moreover, M_F contains all rational expressions in s.

We state without proof two lemmas.

LEMMA 2.1. Let $\{\phi_n\}$ be a sequence of positive functions such that:

(i) $\int_{-\infty}^{\infty} \phi_n(t)dt=1$ for all n and (ii) supp $\phi_n \subseteq [0,\epsilon_n]$, where $\epsilon_n \to 0$

(supp ϕ_n is the closure of the set on which ϕ_n is not zero). Then for $f \in C$,and for each $m > 0$, the sequence $\{\phi_n*f\}$ converges uniformly to f on $[0,m]$.

LEMMA 2.2. Let $\{\phi_n\}$ be a sequence of functions such that on each interval $[0,m]$ $\phi_n \to \phi$ uniformly. Then, for $f \in C$ and each interval $[0,m]$, $\phi_n*f \to \phi*f$ uniformly.

A subset S of a countable union space X is said to be dense if for each $x \in X$ there is some sequence $\{x_n\}$ in S that converges to x.

THEOREM 2.3. C is dense in M_F.

PROOF. Let $x \in M^k$ for some k. Let $\{\phi_n\}$ be a sequence of k-times continuously differentiable positive functions satisfying the following three properties:

(i) $\int_{-\infty}^{\infty} \phi_n(t)dt = 1$ for all n; (ii) supp $\phi_n \subseteq [0,1/n]$, for $n=1,2,...,$

(iii) For each n, $\phi_n^{(j)}(0) = 0$, $j=0,1,2,...,k$. Then, by Lemma 2.1, for any continuous function f, $\{\phi_n*f\}$ converges uniformly to f on compact subsets of $[0,\infty)$. Thus $[(\ell^k*x)*\phi_n^{(k)}]*\ell^k$, $\ell^k*x \in C$ for all n and $[(\ell^k*x)*\phi_n^{(k)}]*\ell^k = (\ell^k*x)*\phi_n \to \ell^k*x$

uniformly on compact subsets of $[0,\infty)$. That is $\{\ell^k * x * \phi_n^{(k)}\}$ converges to x in M_F. Thus, C is dense in M_F.

THEOREM 2.4. The mapping $M_F \to M_F$ given by $x \to x*y$, where $y \in M_F$, is sequentially continuous.

PROOF. Let $\{x_n\}$ be a sequence in M_F that converges to x. Thus, for some k, $\ell^k * x_n, \ell^k * x \epsilon C$ for $n = 1,2,\ldots$, and on each interval $[0,m]$ the sequence $\{\ell^k * x_n\}$ converges uniformly to $\ell^k * x$. Let $y \in M_F$. Then, for some i, $\ell^i * y \in C$. Hence, $\ell^{k+i} * x_n, \ell^{k+i} * x \in C$ for $n = 1,2,\ldots$ and, by Lemma 2.2, the sequence $\{\ell^{k+i} * x_n * y\}$ converges uniformly on each interval $[0,m]$ to $\ell^{k+i} * x * y$. That is, $\{x_n * y\}$ converges to $x*y$ in M_F. This establishes the theorem.

The collection of infinitely smooth complex-valued functions on $(-\infty,\infty)$ will be denoted by C^∞. We now define the product of an infinitely smooth function and an operator in M_F.

DEFINITION 2.5. For $\phi \in C^\infty$ and $x \in M^k$, the product of ϕ and x, denoted by $\phi.x$, is given by

$$\phi.x = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{\phi^{(j)}(\ell^k * x)}{\ell^{k-j}} \qquad (\text{where } f/\ell^0 = f)$$

REMARKS 2.6. (i) If $\phi \in C^\infty$ and $f \in C$, then $\phi.f$ is ordinary multiplication.

(ii) $1.x=x$ for $x \in M_F$.

It is not clear that multiplication is a well-defined operation. To see that it is,notice that if $n > m$, $f/\ell^n = g/\ell^m$ if and only if $f = \ell^{n-m} * g$. Now, let

$$F = \sum_{j=0}^{n+k} (-1)^j \binom{n+k}{j} \ell^j * [\phi^{(j)}(\ell^k * f)] - \sum_{j=0}^{n} (-1)^j \binom{n}{j} \ell^{k+j} * [\phi^{(j)} f],$$

where k is a nonnegative integer. After differentiating F n+k times we obtain $F^{(n+k)}(t) = 0$ for $t > 0$ and $F^{(j)}(0^+) = 0$ for $j = 0,1,2,\ldots,n+k-1$. Thus, $F(t) = 0$ for $t > 0$. Therefore, $\phi.(f/\ell^n) = \phi.(\ell^k * f/\ell^{n+k})$.

THEOREM 2.7. For $\phi,\psi \epsilon C^\infty$ and $x \in M_F$, $(\phi.\psi).x = \phi.(\psi.x)$.

PROOF. Suppose that $\phi,\psi \epsilon C^\infty$ and $x \epsilon M^k$. Then,

$$\phi.(\psi.x) = \sum_{n=0}^{k} (-1)^n \binom{k}{n} \sum_{j=0}^{k-n} (-1)^j \binom{k-n}{j} \frac{\phi^{(j)}[\psi^{(n)}(\ell^k * x)]}{\ell^{k-n-j}}$$

$$= \sum_{n=0}^{k} \sum_{j=0}^{n} (-1)^n \binom{k}{j} \binom{k-j}{n-j} \frac{\phi^{(n-j)}\psi^{(j)}(\ell^k * x)}{\ell^{k-n}} . \qquad (2.1)$$

The last equality follows by the transformation u=j+n and v=n.

Now,

$$(\phi.\psi).x = \sum_{n=0}^{k} (-1)^n \binom{k}{n} \frac{(\phi\psi)^{(n)}(\ell^k * x)}{\ell^{k-n}} . \qquad (2.2)$$

By Leibniz's formula, we see that (2.1) and (2.2) are equivalent. Hence, the theorem is proved.

EXAMPLES 2.8.

(i) $$\phi.s^n = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \phi^{(j)}(0)s^{n-j}, \qquad \text{for } n=0,1,2,\ldots$$

(where $s^0 = \delta$, $\delta = \ell/\ell$ is the identity operator).

(ii) $$t^m.s^n = \begin{cases} 0 & n<m \\ (-1)^m \dfrac{n!}{(n-m)!} s^{n-m} & n \geq m \end{cases}$$

REMARKS. 2.9. (i) Example (i) follows by substituting the formula

$$s^n * \phi = \phi^{(n)} + \sum_{k=0}^{n-1} \phi^{(k)}(0)s^{n-(k+1)} \qquad \text{(see [1])}$$

into the definition for $\phi.s^n$.

(ii) Example (ii) follows by substituting $\phi(t) = t^m$ into Example (i).

THEOREM 2.10. The mapping M_F into M_F given by $x \to \phi.x$, where $\phi \varepsilon C^\infty$, is sequentially continuous.

PROOF. Suppose that $\{x_n\}$ converges to x in M_F. That is, for some k, $\ell^k * x_n$, $\ell^k * x \varepsilon C$ $n = 1,2,\ldots$ and on each interval $[0,m]$, $\{\ell^k * x_n\}$ converges uniformly to $\ell^k * x$.

Now, $$\ell^k * (\phi.x_n) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \ell^j * [\phi^{(j)}(\ell^k * x_n)] \ \varepsilon\ C \quad \text{for } n = 1,2,\ldots .$$

Moreover, for each $0 \leq j \leq k$ and on each interval $[0,m]$, $\{\ell^j * (\phi^{(j)}(\ell^k * x_n))\}$ converges uniformly to $\ell^j * (\phi^{(j)}(\ell^k * x))$. Hence, $\{\phi.x_n\}$ converges to $\phi.x$ in M_F. This establishes the theorem.

If we consider C^∞ as a subspace of M_F, then the mapping from C^∞ into M_F given by $\phi \to \phi.x$ is not sequentially continuous. To see this, let ψ be any infinitely smooth positive function such that: (i) $\psi(0)=1$, (ii) $\psi(t)=0$ for $|t| > 1$, and (iii) $\int_{-\infty}^{\infty} \psi(t)dt=1$. Now, for $n=1,2,\ldots$, let $\phi_n(t) = \psi(nt)$. Then, for each n and for $t > 0$, $0 < (\ell * \phi_n)(t) = \int_0^t \phi_n(u)du = \frac{1}{n} \int_0^{nt} \psi(u)du < 1/n$.

Thus, $\{\phi_n\}$ converges to 0 in M_F. But, for each n, $\phi_n.\delta=\phi_n(0)\delta=\delta$. Hence, $\{\phi_n.\delta\}$ does not converge to 0 in M_F.

In the theory of distributions, the countable family of seminorms $\gamma_{n,m}(\phi) = \sup\{|\phi^{(n)}(t)| : |t|<m\}$, $n=0,1,2,\ldots$, $m=1,2,\ldots$ is used to generate a topology for C^∞. In the following theorem, we assume that C^∞ is given this

topology. The proofs of the next theorem and corollary are straightforward and thus omitted.

THEOREM 2.11. The mapping from C^∞ into M_F given by $\phi \to \phi.x$ is sequentially continuous.

COROLLARY 2.12. Let $\phi(t) = \sum_{n=0}^{\infty} \alpha_n t^n$ for $|t| < \infty$.

Then for $x \in M_F$, $\phi.x = \sum_{n=0}^{\infty} \alpha_n (t^n.x)$.

3. DIFFERENTIAL EQUATIONS AND OPERATIONAL EQUATIONS.

It is known that the only solutions, within the framework of distribution theory, to a linear homogeneous ordinary differential equation with infinitely smooth coefficients are the classical ones. But, when the coefficients have singularities, there may be other distributional solutions (see [5] Littlejohn and Kanwal, and Wiener [6]). In this section we will show that in M_F the situation is similar. For simplicity, we will only consider second order differential and operational equations. However, what follows is also true for nth order equations.

When solving the initial value problem $y'' + \alpha_1 y' + \alpha_2 y = 0$, $y(0) = \beta_1$ and $y'(0) = \beta_2$ using operational methods, the corresponding operational equation is $s^2 * y + \alpha_1 (s*y) + \alpha_2 y = (\beta_2 + \alpha_1 \beta_1)\delta + \beta_1 s$. This follows by the formula given in Remark 2.9(i). By a similar argument, the differential equation $y'' + \phi_1 y' + \phi_2 y = 0$, where $\phi_1, \phi_2 \in C^\infty$, corresponds to the operational equation

$$s^2 * x + \phi_1.(s*x) + \phi_2.x = (\beta_2 + \beta_1 \phi_1(0))\delta + \beta_1 s$$

where $y(0) = \beta_1$ and $y'(0) = \beta_2$. Thus, we will consider second order operational equations of the form $s^2 * x + \phi_1.(s*x) + \phi_2.x = \alpha_1 \delta + \alpha_2 s$, where α_1 and α_2 are constants and $\phi_1, \phi_2 \in C^\infty$.

THEOREM 3.1. Let α_1 and α_2 be constants and $\phi_1, \phi_2 \in C^\infty$. The only solution to the operational equation $s^2 * x + \phi_1.(s*x) + \phi_2.x = \alpha_1 \delta + \alpha_2 s$ is the classical solution to the corresponding differential equation.

PROOF. Let $x \in M^k$ such that

$$s^2 * x + \sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} \frac{\phi_1^{(j)}(\ell^k * x)}{\ell^{k+1-j}} + \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{\phi_2^{(j)}(\ell^k * x)}{\ell^{k-j}}$$
$$= \alpha_1 \delta + \alpha_2 s.$$

By convolving both sides with ℓ^{k+2} we obtain

$$\ell^k * x + \sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} \ell^{j+1} * [\phi_1^{(j)} (\ell^k * x)] \tag{3.1}$$
$$+ \sum_{j=0}^{k} (-1)^j \binom{k}{j} \ell^{j+2} * [\phi_2^{(j)} (\ell^k * x)] = \alpha_1 \ell^{k+2} + \alpha_2 \ell^{k+1}.$$

Now, notice that each term, except the first $\ell^k * x$, in (3.1) is differentiable on $(0,\infty)$. Hence, $\ell^k * x$ is also differentiable on $(0,\infty)$. By an inductive argument, we obtain $\ell^k * x \in C^\infty(0,\infty)$.
After differentiating (3.1) k + 2 times we obtain

$$(\ell^k * x)^{(j)}(0^+) = 0 \quad \text{for } j = 0,1,2,\ldots,k-1, \tag{3.2}$$

$$(\ell^k * x)^{(k)}(0^+) = \alpha_2, \text{ and } (\ell^k * x)^{(k+1)}(0^+) = \alpha_1 - \phi_1(0)\alpha_2.$$

Moreover, $$(\ell^k * x)^{(k+2)} + \phi_1(\ell^k * x)^{(k+1)} + \phi_2(\ell^k * x)^{(k)} = 0.$$

Now let
$$g = (\ell^k * x)^{(k)}. \tag{3.3}$$

Then, g is the unique solution to the initial value problem $g'' + \phi_1 g' + \phi_2 g = 0$, $g(0) = \alpha_2$, and $g'(0) = \alpha_1 - \phi_1(0)\alpha_2$. By (3.2) and (3.3), $\ell^k * x = \ell^k * g$. That is, $x = g$, $g(0) = \alpha_2$, $g'(0) = \alpha_1 - \phi_1(0)\alpha_2$. This establishes the theorem.

The following examples demonstrate that the situation is different when singularities are allowed in the coefficients. By using the formula given in Example 2.8(ii), it is easy to verify that x is a solution to the corresponding operational equation.

EXAMPLES 3.2.

(i) $x = \delta$ is a solution to $t.(s^2 * x) + (2-t).(s * x) - x = 0$.

(ii) $x = \delta - s$ is a solution to $t.(s^2 * x) + (3-t).(s * x) - x = 0$.

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