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PARETO OPTIMALITY FOR NONLINEAR INFINITE DIMENSIONAL CONTROL SYSTEMS

by

EVGENIOS P. AVGERINOS
University of Thessaloniki
School of Technology,
Thessaloniki 54006, GREECE

and

NIKOLAOS S. PAPAGEORGIOU
University of California
1015 Department of Mathematics
Davis, California 95616, U.S.A.

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ABSTRACT. In this note we establish the existence of Pareto optimal solutions for nonlinear, infinite dimensional control systems with state dependent control constraints and an integral criterion taking values in a separable, reflexive Banach lattice. An example is also presented in detail. Our result extends earlier ones obtained by Cesari and Suryanarayana.

KEY WORDS AND PHRASES. Pareto efficient points, Banach lattice, mild solution, evolution operator, compactness, Fatou's lemma, orientor field, nonlinear parabolic equation.

1. INTRODUCTION

In recent years there has been an increasing interest in optimization problems with multiple objectives conflicting with one another. The subject has its origins in mathematical economics and in particular in welfare theory and from there it passed into other subjects like game theory, operations research, optimization and optimal control.

Such problems, in the context of optimal control theory, were recently considered by Cesari and Suryanarayana in a series of interesting papers [5], [6], [7]. We should also mention the earlier work of Olech [9], who, motivated from the fundamental work of Cesari [4], studied similar problems in $\mathbb{R}^n$.

The aim of this note is to extend the finite dimensional existence result for Pareto solutions of Cesari–Suryanarayana [5], to infinite dimensional control systems. Exploiting some recent results on the extended Fatou's lemma, obtained by Balder [1] and Papageorgiou

[11], we are able to prove the closedness of a certain orientor field and through that establish the existence of Pareto optimal solutions.

2. PRELIMINARIES

Let (Ω,Σ) be a measurable space and X a separable Banach space. Throughout this note we will be using the following notations:

$$P_{f(c)}(X) = \{A \subseteq X\colon \text{nonempty, closed, (convex)}\}$$

and $$P_{(w)k(c)}(X) = \{A \subseteq X\colon \text{nonempty, (w–)compact, (convex)}\}.$$

A multifunction (set valued function) $F\colon \Omega \to P_f(X)$ is said to be measurable if and only if for all $y \in X$, $\omega \to d(y, F(\omega)) = \inf\{\|y-x\|\colon x \in F(\omega)\}$ is measurable. If there exists a σ–finite measure $\mu(\cdot)$, with respect to which Σ is complete, then the above definition of measurability is equivalent to saying that $\mathrm{Gr}F = \{(\omega,x) \in \Omega \times X\colon x \in F(\omega)\} \in \Sigma \times B(X)$, with $B(X)$ being the Borel σ–field of X (graph measurability). For further details on measurable multifunctions we refer to the survey paper of Wagner [16]. By S_F^1 we will denote the set of selectors of $F(\cdot)$, that belong in the Lebesgue–Bochner space $L^1(X)$ i.e. $S_F^1 = \{f \in L^1(X)\colon f(\omega) \in F(\omega)$ μ–a.e.$\}$. This set may be empty. However if $F(\cdot)$ is integrably bounded (i.e. $F(\cdot)$ is measurable and $\omega \to |F(\omega)| = \sup\{\|x\|\colon x \in F(\omega)\} \in L_+^1$), then $S_F^1 \neq \emptyset$. Using S_F^1 we can define a set valued integral for $F(\cdot)$ by setting $\int_\Omega F(\omega)\, d\mu(\omega) = \{\int_\Omega f(\omega) d\mu(\omega)\colon f \in S_F^1\}$.

Let Y,Z be Hausdorff topological spaces and let $G\colon Y \to 2^Z\setminus\{\emptyset\}$. We say that $G(\cdot)$ is upper semicontinuous by inclusion (u.s.c.i.), if for all $y_n \to y$ in Y $\overline{\lim}\, G(y_n) = \{z \in Z\colon z = \lim z_{n_k}, z_{n_k} \in G(y_{n_k}), n_1 < n_2 < \ldots\} \subseteq G(y)$.

Finally, in the next section we will be using some notions and results from the theory of ordered vector spaces. For the necessary background we refer to the books of Peressini [14] and Schaefer [15].

3. EXISTENCE THEOREM

Let Y be a locally convex vector space with a partial order induced by a nonempty, closed, convex and pointed cone Y_+. For $y, y' \in Y'$ we write $y \leq y'$ if and only if $y'-y \in Y_+$. Let $A \subseteq Y$. A vector $x_0 \in \overline{A}$ is said to be Pareto efficient for A, if $(x_0-\dot{Y}_+) \cap \overline{A} = \emptyset$, where $\dot{Y}_+ = Y_+\setminus\{0\}$. So the Pareto efficient (or Pareto optimal) points of A, are those points of $\overline{A}$ which are minimal for the partial order induced by Y_+. The set of Pareto efficient points of A will be denoted by $\mathrm{Eff}(A)$.

Recall (see Penot [13] or Peressini [14]), that (Y, Y_+) is (countably) Daniell if every decreasing (sequence) net bounded from below, has an infimum and converges to that infimum. The class of Daniell ordered spaces includes the following ones:

(a) All ordered vector spaces which have compact order intervals (resp. weakly compact, if the order is normal).

(b) All semi–reflexive ordered vector spaces, with normal order.

(c) All ordered vector spaces, with Y_+ complete and having a bounded base B s.t. $0 \notin \overline{B}$ (in particular then, if Y_+ is locally compact).

(d) All (countably) order complete Banach lattices, unless they contain a lattice isomorphic to l^∞.

Note that every (countably) Daniell space is (countably) order complete.

The following existence result is well known among people working in Pareto optimization (see for example Penot [13]).

PROPOSITION: If (Y,Y_+) is a Daniell vector space and $A \subseteq Y$ is nonempty and bounded below, then $\text{Eff}(A) \neq \emptyset$.

Now let $T = [0,b]$, a bounded, closed interval in $\mathbb{R}_+$, X a separable Banach space (the state space), Z another separable Banach space (the control space) and Y a separable, reflexive, order complete, Banach lattice.

We will consider the following infinite dimensional, nonlinear control system:

$$\left\{\begin{array}{l} \dot{x}(t) = A(t)x(t) + f(t,x(t), u(t)) \\ x(0) = x_0,\ u(t) \in U(t,x(t)) \text{ a.e.} \end{array}\right\} (*)$$

By a solution of this system, we will understand a mild (integral) solution. A pair of functions $x(\cdot) \in C(T,X)$ and $u(\cdot) \in L^1(Z)$, that satisfy the dynamic constraints (*), are said to be an "admissible pair". In particular $x(\cdot)$ is an "admissible trajectory", while $u(\cdot)$ is an "admissible control". We will denote the set of admissible pairs by $A(x_0)$. Finally note that system (*) has feedback type constraints, since the multifunction $U(\cdot,\cdot)$ depends also on the state.

To this control system, we associate a Y–valued cost criterion of the following form:

$$J(x,u) = \int_0^b L(t,x(t), u(t))\, dt$$

with $L: T \times X \times Z \to Y$.

Our goal is to prove a theorem saying that every vector in $\text{Eff}(J(A(x_0)))$ is realized by an admissible pair.

To this end, we need the following set of hypotheses on the data of the problem.

<u>H(A)</u>: $\{A(t): t \in T\}$ are linear, unbounded operators on $D(A(t)) \subseteq X$, that generate a strongly continuous evolution operator $S(t,s) \in \mathscr{L}(X)$, $0 \leq s \leq t \leq b$, which is compact for $t-s > 0$.

<u>H(f)</u>: $f: T \times X \times Z \to X$ is a function s.t.

(1) $t \to f(t,x,u)$ is measurable,

(2) $(x,u) \to f(t,x,u)$ is sequentially continuous from $X \times Z_w$ into X_w (where X_w,

Z_w denote the Banach spaces X, Z with their respective weak topologies),

(3) $\|f(t,x,u)\| \leq a(t) + b(t)\,(\|x\| + \|u\|)$ a.e. with $a(\cdot), b(\cdot) \in L^1_+$.

H(L): L: $T \times X \times Z \to Y$ is a measurable function.

H(Q): For every $(t,x) \in T \times X$, the set $Q(t,x) = \{(v,\eta) \in X \times Y: v = f(t,x,u),\ u \in U(t,x),\ L(t,x,u) \leq \eta\}$ is convex and $x \to Q(t,x)$ is u.s.c.i. from X into X_w.

H(U): U: $T \times X \to P_{fc}(Z)$ is a multifunction s.t.

(1) $(t,x) \to U(t,x)$ is measurable,

(2) $x \to U(t,x)$ is u.s.c.i. from X into Z_w,

(3) $U(t,x) \subseteq W$ a.e., where $W \in P_{wkc}(Z)$.

H_a: $A(x_0) \neq \emptyset$ (i.e. there exist admissible "state–control" pairs).

H_b: $J(A(x_0))$ is order bounded in X.

Hypothesis H_a is a controllability type hypothesis, while hypothesis H_b is satisfied if for example $|L(t,x,u)| \leq a'(t) + b'(t)\,(\|x\| + \|u\|)$ a.e. with $a'(\cdot), b'(\cdot) \in L^1(Y_+)$. Recall $|y| = y^+ + y^-$.

THEOREM 1: If hypotheses H(A), H(f), H(L), H(Q), H(U), H_a and H_b hold, then $Eff(J(A(x_0))) \neq \emptyset$ and every element in this set can be realized by an admissible "state–control" pair.

PROOF: Recalling that a reflexive Banach lattice is a Daniell space and using hypothesis H_b and the proposition, we deduce that $Eff(J(A(x_0))) \neq \emptyset$.

Let $e \in Eff(J(A(x_0)))$. Then by definition $e \in \overline{J(A(x_0))}$. So there exist $y_k \in J(A(x_0))$, $k \geq 1$ s.t. $y_k \xrightarrow{s} e$. We have $y_k = J(x_k, u_k)$, with $(x_k, u_k) \in A(x_0)$ for all $k \geq 1$.

Our claim is that $\{x_k\}_{k\geq 1}$ is relatively compact in $C(T,X)$. To this end, we will first determine an a priori bound for the trajectories of (*). So let $x(\cdot) \in C(T,X)$ be such a trajectory. We have

$$x(t) = S(t,0)\,x_0 + \int_0^t S(t,s)\,f(s,x(s)),\,u(s))\,ds,\quad t \in T,\quad u(\cdot) \in S^1_{U(\cdot,x(\cdot))}$$

$$\Longrightarrow \quad \|x(t)\| \leq \|S(t,0)x_0\| + \int_0^t \|S(t,s)\|.\|\,f(s,x(s),\,u(s))\|\,ds$$

$$\Longrightarrow \quad \|x(t)\| \leq M\,\|x_0\| + M\int_0^t [a(s) + b(s)(\|x(s)\| + \|u(s)\|)]ds \ (\text{since } \|S(t,s)\| \leq M)$$

$$\Longrightarrow \quad \|x(t)\| \leq M\,(\|x_0\| + \|a\|_1 + |W|\,.\,\|b\|_1) + M\int_0^t b(s)\,\|x(s)\|\,ds$$

$$(|W| = \sup\{|u|: u \in W\})$$

Invoking Gronwall's inequality, we get that $\|x(t)\| \leq M_1$ for all $t \in T$ and all trajectories $x(\cdot)$ of (*). Next let $t, t' \in T$, $t \leq t'$. For any $(x,u) \in N = \{(x_k, u_k)\}_{k\geq 1} \subseteq A(x_0)$ we have

$$
\begin{aligned}
&\|x(t') - x(t)\| \\
&= \|S(t',0)\,x_0 + \int_0^{t'} S(t', s)\, f(s,x(s), u(s))\, ds - S(t,0)\, x_0 - \int_0^t S(t,s)\, f(s,x(s), u(s))\, ds\| \\
&\le \|S(t',0)\, x_0 - S(t,0)\, x_0\| + \int_t^{t'} \|S(t',s)\| . \|f(s,x(s), u(s))\|\, ds \\
&\qquad + \int_0^t \|S(t',s) - S(t,s)\| \, . \, \|f(s,x(s), u(s))\|\, ds
\end{aligned}
$$

Let us estimate the three summands in the right hand side of the above inequality. Because of the strong continuity of the evolution operator, given $\epsilon > 0$ there exists $\delta_1 > 0$ s.t. for $|t'-t| < \delta_1$, we have

$$\|S(t',0)\, x_0 - S(t,0)\, x_0\| < \epsilon/3 \tag{1}$$

Also $\int_t^{t'} \|S(t',s)\| \, \|f(s,x(s), u(s))\|\, ds \le M \int_t^{t'} [a(s) + b(s)\,(M_1 + |W|)]ds$. Since $a(\cdot)$, $b(\cdot) \in L^1_+$, we can find $\delta_2 > 0$ s.t. for $|t'-t| < \delta_2$, we have

$$M \int_t^{t'} [a(s) + b(s)\,(M_1 + |W|)]ds < \epsilon/3 \tag{2}$$

Finally for $\epsilon_1 > 0$ we have

$$
\begin{aligned}
&\int_0^t \|S(t',s) - S(t,s)\| \, . \, \|f(s,x(s), u(s))\|\, ds \\
\le\; &\int_0^{t-\epsilon_1} \|S(t',s) - S(t,s)\| \, . \, \|s(s,x(s), u(s))\|\, ds + \int_{t-\epsilon_1}^t 2M\,(a(s) + b(s)\,(M_1 + |W|))ds
\end{aligned}
$$

Let $\epsilon_1 > 0$ be such that $\int_{t-\epsilon_1}^t 2M(a(s) + b(s)\,(M_1 + |W|))ds < \epsilon/6$. Also from proposition 2.1 of [11], we know that because of the compactness hypothesis on $S(t,s)$ for $t-s > 0$ (see hypothesis $H(A)$), we have that $t \to S(t,s)$ is continuous in the operator norm, uniformly for all s s.t. $t-s$ is bounded away from zero. Thus we can find $\delta_3 > 0$ s.t. for $|t'-t| < \delta_3$ we have $\|S(t',s) - S(t,s)\| < \epsilon/6(\|a\|_1 + \|b\|_1\,(M_1 + |W|))$ for all $s \in T$ with $t-s \ge \epsilon_1$. So we get

$$
\begin{aligned}
&\int_0^{t-\epsilon_1} \|S(t',s) - S(t,s)\| \, . \, \|f(s,x(s), u(s))\|\, ds < \epsilon/6 \\
\Longrightarrow\quad &\int_0^t \|S(t',s) - S(t,s)\| \, . \, \|f(s,x(s), u(s))\|\, ds < \epsilon/3
\end{aligned} \tag{3}
$$

From (1), (2) and (3) above, we have that for $|t'-t| < \delta = \min\{\delta_1, \delta_2, \delta_3\}$, we have $\|x(t') - x(t)\| < \epsilon$ for all $x \in \{x_k\}_{k\ge1}$. Thus the sequence $\{x_k\}_{k\ge1}$ is equicontinuous.

Next observe that for all $k\ge1$ and all $t \in T$

$$x_k(t) \in S(t,0)\, x_0 + \int_0^t \overline{S(t,s)\ V(s)}\, ds$$

where $V(s) = \{x \in X: \|x\| \leq a(s) + b(s)\,(M_1 + |W|)\}$. So $V(s)$ is almost everywhere bounded, closed and since $S(t,s)$ is compact for $t-s > 0$, $\overline{S(t,s)\ V(s)}$ is almost everywhere compact and clearly measurable in s. So by Radström's embedding theorem (see for example [11]), we get that $\int_0^t \overline{S(t,s)\ V(s)}\, ds \in P_{kc}(X), \Longrightarrow \{x_k(t)\}_{k\geq 1} \in P_{kc}(X)$. So invoking the Arzela–Ascoli theorem, we deduce that $\{x_k(\cdot)\}_{k\geq 1}$ is relatively compact in $C(T,X)$.

Observe that $\{u_k\}_{k\geq 1} \subseteq S^1_W$ and the latter is w–compact in $L^1(Z)$ (see proposition 3.1 of [10]), and by the Eberlein–Smulian theorem is sequentially w–compact. So by passing to a subsequence if necessary, we may assume that $x_k \to x$ in $C(T,X)$ and $u_k \xrightarrow{w} u$ in $L^1(Z)$. Set $v_k(t) = f(t, x_k(t), u_k(t))$ and $\eta_k(t) = L(t,x_k(t), u_k(t))$. Note that $\int_0^b \eta_k(s)\, ds = J(x_k, u_k) \xrightarrow{s} e$. Applying theorem 2.1 of Balder [2], we get $\hat{\eta} \in L^1(Y)$ s.t. $\int_0^b \hat{\eta}(t)\, dt = e$ and $\hat{\eta}(t) \in \overline{\text{conv}}\ \text{w–}\overline{\lim}\ \{\eta_k(t)\}_{k\geq 1}$ a.e.. Also note that $v_k \in S^1_{K(\cdot)}$ where $K(t) = \overline{\text{conv}}\ f(t,x_k(t), u_k(t))$. But because of the compactness of $\{\overline{x_k(t)}\}_{k\geq 1}$ in X and since $\overline{\{u_k(t)\}}^{w}_{k\geq 1} \subseteq W$ and $f(t,\cdot,\cdot)$ is continuous on $X \times Z_w$ into X_w, we deduce that $K(t) \in P_{wkc}(X)$ a.e. and clearly is measurable. So proposition 3.1 of [10] tells us that $S^1_{K(\cdot)}$ is sequentially w–compact in $L^1(X)$. Hence we may assume that $v_k \xrightarrow{w} v$ in $L^1(X)$ and furthermore from theorem 3.1 of [12], we have $v(t) \in \overline{\text{conv}}\ \text{w–}\overline{\lim}\ \{v_k(t)\}_{k\geq 1}$ a.e.. Thus we have

$$(v(t), \eta(t)) \in \overline{\text{conv}}\ \text{w–}\overline{\lim}\ (v_k(t), \eta_k(t)) \text{ a.e.}$$

$$\Longrightarrow \quad (v, \eta) \in S^1_{\overline{\text{conv}}\ \text{w–}\overline{\lim}\ Q(\cdot,x_k(\cdot))}$$

But by hypothesis $H(Q)$, we know that $x \to Q(t,x)$ is u.s.c.i. from X into $X_w \times Y_w$. Hence $\text{w–}\overline{\lim}\ Q(t, x_k(t)) \subseteq Q(t,x(t))$ a.e. $\Longrightarrow (v,\eta) \in S^1_{Q(\cdot,x(\cdot))}$.

Let $R(t) = \{u \in U(t,x(t)): v(t) = f(t,x(t), u), L(t,x(t), u) \leq \eta(t)\}$. Since $(v,\eta) \in S^1_{Q(\cdot,x(\cdot))}$, we see that for all $t \in T_0$, $\lambda(T \backslash T_0) = 0$ (λ being the Lebesgue measure on $\mathbb{R}$), we have $R(t) \neq \emptyset$. Set $h_1(t,u) = v(t) - f(t,x(t), u)$ and $h_2(t,u) = \eta(t) - L(t,x(t), u)$. Because of hypotheses $H(f)$ (1), (2) and $H(L)$, we deduce that both $h_1(\cdot,\cdot)$ and $h_2(\cdot,\cdot)$ are measurable. On $(T \backslash T_0) \times Z$ set $h_1(t,u) = 0$ in X and $h_2(t,u) = 0$ in Y. Also let $p: T \times Z \to T \times X \times Z$ be defined by $p(t,u) = (t,x(t), u)$. Clearly this is measurable. So because of hypothesis $H(U)$, $p^{-1}(GrU) \in B(T) \times B(Z)$. But observe that $p^{-1}(GrU) = \{(t,u): u \in U(t,x(t))\} = GrU(\cdot,x(\cdot)) \Longrightarrow GrU(\cdot,x(\cdot)) \in B(T) \times B(Z) \Longrightarrow GrR \in GrU(\cdot,x(\cdot)) \cap \{(t,u) \in T \times Z:$

$h_1(t,u) = 0,\ h_2(t,u) = 0\} \in B(T)\times B(Z)$. Apply Aumann's selection theorem to get $u: T \to Z$ measurable s.t. $u(t) \in R(t)$ for all $t \in T \Longrightarrow u(t) \in Q(t,x(t))$ a.e. $v(t) = f(t,x(t), u(t))$ a.e. and $L(t,x(t), u(t)) \le \eta(t)$ a.e. . Recall that $x_k(t) = S(t,0)\, x_0 + \int_0^t S(t,s)\, v_k(s)\, ds$ and $x_k \to x$ in $C(T,X)$, while $\int_0^t S(t,s)\, v_k(s)\, ds \xrightarrow{w} \int_0^t S(t,s)v(s)\, ds$. So in the limit as $k\to\infty$ $x(t) = S(t,0)\, x_0 + \int_0^t S(t,s)\, f(s,x(s), u(s))\, ds$, $u(t) \in U(t,x(t))$ a.e. i.e. (x,u) is an admissible "state control" pair and $J(x,u) = \int_0^b L(t,x(t), u(t))\, dt \le \int_0^b \eta(t)\, dt \le e$ $\Longrightarrow J(x,u) = e$.

Q.E.D.

REMARKS: (1) Our results extends the finite dimensional work of Cesari–Suryanarayana [3] (theorem 4.1), as well as the infinite dimensional ones by the same authors [6] (theorem 1) and [7] (theorem 8.1), where Z was reflexive, int $Z_+ \ne \emptyset$ and the overall hypotheses on the data were more restrictive.

(2) If $f(t,x,\cdot)$ is linear (i.e. $f(t,x)u$), $u \to L(t,x,u)$ is Y_+–convex (i.e. $L(t,x, \lambda u_1 + (1-\lambda)u_2) \le \lambda\, L(t,x,u_1) + (1-\lambda)\, L(t,x,u_2)$, $\lambda \in [0,1]$, $u_1, u_2 \in Z$) and $(x,u) \to L(t,x,u)$ is scalarly sequentially l.s.c. on $X\times Z_w$ (i.e. for every $y^* \in Y_+^*$ $(x,u) \to (y^*, L(t,x,u))$ is sequentially l.s.c. on $X\times Z_w$ into $\mathbb{R}$), then we claim that $H(Q)$ is satisfied. It is easy to see that because of the above hypotheses $Q(t,x)$ is closed, convex for all $(t,x) \in T\times X$. Also if $x_n \to x$ in X and $(v,\eta) \in w\text{–}\overline{\lim}\, Q(t, x_n)$, then by definition we can find $(v_k, \eta_k) \in Q(t, x_{n_k})$ s.t. $(v_k, \eta_k) \xrightarrow{w\times w} (v,\eta)$. We have

$$v_k = f(t,x_k,u_k),\ u_k \in U(t,x_k) \subseteq W \text{ and } L(t, x_k, u_k) \le \eta_k\ k\ge 1.$$

By passing to a subsequence if necessary, we may assume that $u_k \xrightarrow{w} u$. Then $u \in w\text{–}\overline{\lim}\, U(t,x_k) \subseteq U(t,x)$, $f(t, x_k, u_k) \xrightarrow{w} f(t,x,u) \Longrightarrow v = f(t,x,u)$ and for all $y^* \in Y_+^*$ $(y^*, L(t,x,u)) \le \underline{\lim}\, (y^*, L(t,x_k, u_k)) \le (y^*, \eta) \Longrightarrow L(t,x,u) \le \eta \Longrightarrow (v,\eta) \in Q(t,x) \Longrightarrow Q(t,\cdot)$ is u.s.c.i. .

4. AN EXAMPLE

Let $T = [0,b]$ and V a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial V = \Gamma$. On $T\times V$ we consider the following nonlinear parabolic optimal control problem with vector valued cost functional.

$$\left\{\begin{array}{l} J(x,u) = \int_0^b \int_V L(t,v,x(t,v),\ u(t,v))\, dvdt \to \inf \\ \text{s.t. } \frac{\partial x(t,v)}{\partial t} - \Delta x(t,v) = f(t,v,x(t,v))\, u(t,v) \\ \quad x(t,v) = 0 \qquad \text{on } T\times\Gamma \\ \quad x(0,v) = x_0(v) \qquad \text{on } \{0\}\, xV \\ \quad \|u(t,\cdot)\|_{L^2(V)} \le r \end{array}\right\} \qquad (**)$$

We will need the following hypotheses on the data of (**).

<u>H(f)'</u>: f: $T\times V\times\mathbb{R} \to \mathbb{R}$ is a map s.t.

(1) $(t,v) \to f(t,v,x)$ is measurable

(2) $x \to f(t,v,x)$ is continuous,

(3) $|f(t,v,x)| \le a(t,v)$ a.e., with $a(t,\cdot) \in L^\infty(V)$, $t \to \|a(t,\cdot)\|_{L^\infty(V)}$ belonging in $L^1(T)$.

<u>H(L)'</u>: L: $T\times V\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}^m$ is a function s.t.

(1) $(t,v) \to L(t,v,x,u)$ is measurable

(2) for each $k\ge 1$ $(x,u) \to L_k(t,v,x,u)$ is l.s.c. and convex in u, where $L = (L_k)_{k=1}^m$,

(3) $\phi_k(t,v) \le L_k(t,v,x,u) \le \psi_{1k}(t,v) + \psi_{2k}(t,v)\,(|x|^2 + |u|^2)$, where $\psi_{1k}(\cdot,\cdot) \in L^1(T\times V)$, $\psi_{2k}(t,\cdot) \in L^\infty(V)$ and $t \to \|\psi_{2k}(t,\cdot)\|_{L^\infty(V)} \in L_+^1$.

<u>H_0</u>: $x_0(\cdot) \in L^2(V)$.

Let $X = L^2(V)$, $Z = L^2(V)$ and $Y = \mathbb{R}^m$. Define $\hat{f}: T \times X \to X$ to be the Nemitsky operator corresponding to $f(t,v,x)$ i.e. $\hat{f}(t,x)(\cdot) = f(t,\cdot,x(\cdot))$. Then $[\hat{f}(t,x)\, u](v) = f(t,v,x(v))\, u(v)$. First let us check that indeed $\hat{f}(t,x)\, u(\cdot) \in X = L^2(V)$. We have

$$\int_V |\hat{f}(t,x)\, u(v)|^2\, dv = \int_V |f(t,v,x(v))|^2\, |u(v)|^2\, dv$$

$$\le \int_V a(t,v)^2\, |u(v)|^2\, dv \le \|a(t,\cdot)\|^2_{L^\infty(V)} \cdot \|u\|^2_{L^2(V)}$$

$$\Longrightarrow \quad \|\hat{f}(t,x)\|_{L^2(V)} < \infty.$$

Next let $h \in L^2(V)$. We have

$$(h, \hat{f}(t,x)u) = \int_V h(v)\, f(t,v,x(v))\, u(v)\, dv$$

From Fubini's theorem we know that $t \to \int_V h(v)\, f(t,v,x(v))\, u(v) = (h, \hat{f}(t,x)\, u)$ is measurable $\Longrightarrow t \to \hat{f}(t,x)\, u$ is weakly measurable and since $L^2(V)$ is separable, by Pettis' theorem $t \to \hat{f}(t,x)u$ is strongly measurable. So $\hat{f}(t,x)u$ satisfies hypothesis H(f).

Next let $A = \Delta$ with $D(A) = H_0^1(V) \cap H^2(V)$. It is well known (see for example Barbu [3]) that A generates a semigroup of contractions $S(t)\colon L^2(V) \to L^2(V)$ which are compact for $t > 0$. So we have satisfied hypothesis $H(A)$.

Let $\hat{L}\colon T\times X\times Y \to \mathbb{R}^m$ be defined by $\hat{L}(t,x,u) = \int_V L(t,v,x(v), u(v))\, dv$. Then $\hat{L} = (\hat{L}_k)_{k=1}^m$ and for each $k \in \{1, \ldots, m\}$ $\hat{L}_k(t,x,u) = \int_V L_k(t,v,x(v), u(v))\, dt$. Let $L_k^m\colon T\times V\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be Caratheodory integrands s.t. $\phi_k(t,v) \le L_k^m(t,v,x,u) \le m$ and $L_k^m \uparrow L_k$ as $m\to\infty$ for each $k \in\{1, \ldots, m\}$. This is possible since $L_k(\cdot,\cdot,\cdot,\cdot)$ is a normal integrand (see for example lemma 2 in Balder [1]). Set $\hat{L}_k^m(t,x,u) = \int_V L_k^m(t,v,x(v), u(v))\, dv$. From Fubini's theorem $t \to \hat{L}_k^m(t,x,u)$ is measurable, while $(x,u) \to \hat{L}_k^m(t,x,u)$ is clearly continuous. So $(t,x,u) \to \hat{L}_k^m(t,x,u)$ is measurable and since by the monotone convergence theorem $\hat{L}_k^m \uparrow \hat{L}_k$, we conclude $\hat{L}_k(\cdot,\cdot,\cdot)$ is measurable. So we have satisfied hypothesis $H(L)$. In fact $L(t,x,\cdot)$ is $\mathbb{R}_+^m$–convex (see hypothesis $H(L)'(2)$) and from theorem 5 of Balder [1], (see also Ekeland–Temam [8]), we get that $L(t,\cdot,\cdot)$ is scalarly l.s.c. on $L^2(V) \times L^2(V)_w$. So $H(Q)$ is satisfied (see remark (2) in section 3).

Let $U(t,x) = W = \{u \in L^2(V)\colon \|u\|_{L^2(V)} \le r\}$. Then we have satisfied $H(U)$. Also assume that:

$\underline{H_a'}$: There exists $(x(\cdot,\cdot), u(\cdot,\cdot))$ admissible pair for (**) s.t. $J(x,u) < \infty$.

This assumption implies that H_a is satisfied. Finally note that because of $H(L)'(3)$, H_b is satisfied too. Rewrite (**) in the following abstract form:

$$\left\{\begin{array}{l} \hat{J}(x,u) = \int_0^b \hat{L}(t,x(t), u(t))\, dt \to \inf \\ \text{s.t. } \dot{x}(t) = Ax(t) + \hat{f}(t,x(t))\, u(t) \\ \qquad x(0) = \hat{x}_0 \\ \qquad u(t) \in W \text{ a.e.} \end{array}\right\} \quad (**)'$$

This is a special case of the optimal control problem studied in section 3. So we can invoke theorem 1 and get the following existence result.

THEOREM 2: <u>If</u> hypotheses $H(f)'$, $H(L)'$, H_0 and H_a' hold <u>then</u> $\mathrm{Eff}(A(x_0)) \ne \emptyset$ and the Pareto efficient points of (**) are realized by admissible "state–control" pairs of (**).

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