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IRRESOLUTE MULTIFUNCTIONS

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ABSTRACT. This paper considers a new class of multifunctions, the irresolute multifunctions. For the irresolute multifunctions we give some theorems of characterizations. Some relations between continuous multifunctions and irresolute multifunctions are established.

KEY WORDS AND PHRASES. Quasicontinuous multifunction, irresolute multifunction, strongly continuous multifunction.
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1. INTRODUCTION.

In [1] Levine defines a set A in a topological space X to be semi-open if there exists an open set $U \subset X$ such that $U \subset A \subset \mathrm{Cl}\ U$, where Cl U denotes the closure of U.

The family of all semi-open sets in X is denoted by SO(X). A set is semi-closed if its complement is semi-open. The intersection of all the semi-closed sets containing a set A is the semi-closure of A denoted by Scl A. Also, $\mathrm{Scl}(A)= \mathrm{Scl}(\mathrm{Scl}\ A)$, $A \subset B$ implies $\mathrm{Scl}\ A \subset \mathrm{Scl}\ B$, $A \subset \mathrm{Scl}\ A \subset \mathrm{Cl}\ A$ and that A is semi-closed iff $A=\mathrm{Scl}\ A$ [2], [3].

The notion of irresolute functions was introduced by Crossley and Hildebrand in [4] in this way:

DEFINITION 1. Let X and Y be two topological spaces. A function $f:X\to Y$ is irresolute if for each $V \in SO(Y)$, $f^{-1}(V) \in SO(X)$.

The notion of upper (lower) irresolute multifunctions was introduced by Ewert and Lipski in [5].

DEFINITION 2. Let X and Y be two topological spaces.

(a) A multifunction $F:X\to Y$ is upper irresolute (u.i.) at a point $x \in X$ if for any semi-open set $W \subset Y$ such that $F(x) \subset W$, there exists a semi-open set $U \subset X$ containing x such that $F(U) \subset W$.

(b) A multifunction $F:X\to Y$ is lower irresolute (l.i.) at a point $x \in X$ if or any semi-open set $W \subset Y$ such that $F(x) \cap W \neq \phi$ there is a semi-open set $U \subset X$ containing x such that $F(y) \cap \neq \phi$, $\forall\ y \in U$.

(c) A multifunction $F:X\to Y$ is upper (lower) irresolute if it has this property in any point $x \in X$ [5].

Some properties of the lower (upper) irresolute multifunctions are studied in [5].

The notion of quasicontinuous multifunctions was introduced and studied by Banzaru and Crivat in [6].

DEFINITION 3. Let X and Y be two topological spaces. A multifunction $F:X\to Y$ is quasicontinuous at a point $x \in X$ if for any neighborhood U of x and for any open sets G_1, $G_2 \subset Y$ such that $F(x) \subset G_1$ and $F(x) \cap G_2 \neq \phi$ there exists a non-empty open set $G_U \subset U$ such that $F(G_U) \subset G_1$ and $F(y) \cap G_2 \neq \phi, \forall y \in G_U$.

The multifunction $F:X\to Y$ is quasicontinuous if it has this property at any point $x \in X$ [6].

Some properties of quasicontinuous multifunctions are studied in [7], [6] and [8].

DEFINITION 4. Let X and Y two topological spaces. A multifunction $F:X\to Y$ is irresolute at a point $x\in X$ if for any semi-open sets G_1, $G_2 \subset Y$ such that $F(x) \subset G_1$ and $F(x) \subset G_2 \neq \phi$ there exists a semi-open set $U \subset X$ containing x such that $F(U) \subset G_1$ and $F(y) \cap G_2 \neq \phi, \forall y \in U$.

The multifunction $F:X\to Y$ is irresolute if it has this property at any point $x \in X$.

REMARK 1. If $F:X\to Y$ is irresolute then F is upper and lower irresolute.

REMARK 2. By Theorem 1.1 [8] it follows that if $F:X\to Y$ is irresolute then F is quasicontinuous.

2. CHARACTERIZATIONS.

Let X,Y be two topological spaces and let S(y) and K(y) be classes of all non-empty and non-empty compact subsets of Y, respectively. For a multifunction $F:X\to Y$ we will denote

$$F^+(B) = \{x \in X : F(x) \subset B\}; \; F^-(B) = \{x \in X : F(x) \cap B \neq \phi\}$$

for any subset $B \subset Y$.

DEFINITION 6. Let A be a set of a topological space X.U is a semi-neighbourhood which intersects A if there exists a semi-open set $V \subset X$ such that $V \subset U$ and $V \cap A \neq \phi$.

THEOREM 1. For a multifunction $F:X\to Y$ the following are equivalent:

1. F is irresolute at $x \in X$.
2. For any semi-open sets G_1, $G_2 \subset Y$ with $F(x) \subset G_1$ and $F(x) \cap G_2 \neq \phi$, there results the relation

$$x \in \mathrm{Cl}\,\{\mathrm{Int}\,[F^+(G_1) \cap F^-(G_2)]\}.$$

3. For every semi-open set. G_1, $G_2 \subset Y$ with $F(x) \subset G_1$ and $F(x) \cap G_2 \neq \phi$ and for any open set $U \subset X$ containing x, there exists a non-empty open set $G_u \subset U$ such that $F(G_U) \subset G_1$ and $F(y) \cap G_2 \neq \phi$, $\forall$ $y \in G_U$.

PROOF. (1) => (2). Let G_1, $G_2 \in SO(Y)$ with $F(x) \subset G_1$ and $F(x) \cap G_2 \neq \phi$. Then there is $U \in SO(X)$ containing x such that $F(U) \subset G_1$ and $F(y) \cap G_2 \neq \phi$, $\forall$ $y \in U$, thus

$x \in U \subset F^+(G_1)$ and $x \in U \subset F^-(G_2)$. Then $x \in U \subset F^+(G_1) \cap F^-(G_2)$. Since U is a semi-open set in X, then by Theorem 1 [1] $x \in U \subset Cl\,[Int\,U] \subset Cl\{Int\,[F^+(G_1) \cap F^-(G_2)]\}$.

(2) => (3). Let $G_1, G_2 \in SO(Y)$ be with $F(x) \subset G_1$ and $F(x) \cap G_2 \neq 0$. Then $x \in Cl\{Int[F^+(G_1) \cap F^-(G_2)]\}$. Let $U \subset X$ be any open set such that $x \in U$. Then $U \cap [Int\, F^+(G_1) \cap F^-(G_2)] \neq 0$. Since $Int[F^+(G_1) \cap F^-(G_2)] \subset Int\, F^+(G_1) \cap Int\, F^-(G_2)$ then $U \cap [Int\, F^+(G_1) \cap Int\, F^-(G_2)] \neq 0$. Put $G_U = [Int\, F^+(G_1) \cap Int\, F^-(G_2)] \cap U$, then $G_U \neq 0$, $G_U \subset U$, $G_U \subset Int\, F^+(G_1) \subset F^+(G_1)$ and $G_U \subset Int\, F^-(G_2) \subset F^-(G_2)$ and thus $F(G_U) \subset G_1$ and $F(y) \cap G_2 \neq 0, \forall\, y \in G_U$.

(3) => (1). Let U_x be the system of the open sets from X containing x. For any open set $U \subset X$ such that $x \in U$ and for every semi-open set $G_1, G_2 \subset Y$ with $F(x) \subset G_1$ and $F(x) \cap G_2 \neq 0$, there exists a non-empty open set $G_U \subset U$ such that $F(G_U) \subset G_1$ and $F(y) \cap G_2 \neq 0, \forall\, y \in G_U$. Let $W = \bigcup_{U \in U_x} G_U$, then W is open, $x \in cl\, W$, $F(W) \subset G_1$ and $F(z) \cap G_2 \neq 0$, $\forall\, z \in W$. Put $S = W \cup \{x\}$, then $W \subset S \subset Cl\, W$, thus W is a semi-open set in X, $x \in S$, $F(S) \subset G_1$ and $F(t) \cap G_2 \neq 0, \forall\, t \in S$, thus F is irresolute at x.

THEOREM 2. For a multifunction $F{:}X \to Y$ the following are equivalent:

1. F is irresolute.
2. For every semi-open set $G_1, G_2 \subset Y$, $F^+(G_1) \cap F^-(G_2) \in SO(X)$.
3. For every semi-closed set $V_1, V_2 \subset Y$, $F^-(V_1) \cup F^+(V_2)$ is a semi-closed set in X.
4. For every set $B_1, B_2 \subset Y$, there results the relation
$$Int\{Cl[F^-(B_1) \cup F^+(B_2)]\} \subset F^-(Scl\, B_1) \cup F^+(Scl\, B_2).$$
5. For every sets $B_1, B_2 \subset Y$, there results the relation
$$Scl[F^-(B_1) \cup F^+(B_2)] \subset F^-(Scl\, B_1) \cup F^+(Scl\, B_2).$$
6. For every set $B_1, B_2 \subset Y$, there results the relation
$$sInt[F^-(B_1) \cap F^+(B_2)] \supset F^-(sInt\, B_1) \cap F^+(sInt\, B_2).$$
7. For each point x of X and for each semi-neighbourhood V_1 of $F(x)$ and for each semi-neighbourhood V_2 which intersects $F(x)$, $F^+(V_1) \cap F^-(V_2)$ is a semi-neighbourhood of x.
8. For each point x of X and for each semi-neighbourhood V_1 of $F(x)$ and for each semi-neighbourhood V_2 which intersects $F(x)$, there is a semi-neighbourhood U of x such that $F(U) \subset V_1$ and $F(y) \cap V_2 \neq 0$, $\forall\, y \in U$.

PROOF. (1) => (2). Let G_1, $G_2 \in SO(Y)$ and $x \in F^+(G_1) \cap F^-(G_2)$, thus $F(x) \subset G_1$ and $F(x) \cap G_2 \neq 0$, then F being irresolute according to the Theorem 1, implication (1) => (2) there follows that $x \in Cl\{Int[F^+(G_1) \cap F^-(G_2)]\}$ and as x is choosen arbitrarily in $F^+(G_1) \cap F^-(G_2)$, there follows that $F^+(G_1) \cap F^-(G_2) \subset Cl\{Int[F^+(G_1) \cap F^-(G_2)]\}$ and thus $F^+(G_1) \cap F^-(G_2)$ is a semi-open set by Theorem 1 of [6].

(2) => (3). For if $V \subset Y$, then $F^-(Y-V)=X-F^+(V)$ and $F^+(Y-V) = X-F^-(V)$.

(3) => (4). Suppose that (3) holds and let B_1, B_2 two arbitrary subsets of Y, then Scl B_1 and Scl B_2 are semi-closed sets in Y. Then $F^-(\text{Scl } B_1) \cup F^+(\text{Scl } B_2)$ is a semi-closed set of X. By Theorem 1 of [3]

$$Int\{Cl\ [F^-(\text{Scl } B_1) \cup F^+(\text{Scl } B_2)]\} \subset F^-(\text{Scl } B_1) \cup F^+(\text{Scl } B_2).$$

Since we have $A \subset \text{Scl } A$ then $F^+(A) \subset F^+(\text{Scl } A)$ and $F^-(A) \subset F^-(A) \subset F^-(\text{Scl } A)$. Consequently,

$$Int\{Cl\ [F^-(B_1) \cup F^+(B_2)]\} \subset Int\{Cl[F^-(\text{Scl } B_1) \cup F^+(\text{Scl } B_2)]\} \subset$$

$$\subset F^-(\text{Scl } B_1) \cup F^+(\text{Scl } B_2).$$

(4) => (5). From $\text{Scl } A = A \cup Int\ Cl\ A$ follows $\text{Scl}[F^-(B_1) \cup F^+(B_2)] = [F^-(B_1) \cup F^+(B_2)] \cup Int\{Cl\ [F^-(B_1) \cup F^+(B_2)]\} \subset [F^-(B_1) \cup F^+(B_2)] \cup F^-(\text{Scl } B_1) \cup F^+(\text{Scl } B_2) \subset F^-(\text{Scl } B_1) \cup F^+(\text{Scl } B_2)$.

(5) => (6) $X - \text{sInt}[F^-(B_1) \cap F^+(B_2)] = \text{Scl } [X-F^-(B_1) \cap F^+(B_2)] = \text{Scl } [(X-F^-(B_1)) \cup (X-F^+(B_2))] = \text{Scl } [F^+(Y-B_1) \cup F^-(Y-B_2)] \subset F^+(\text{Scl}(Y-B_1)) \cup F^-(\text{Scl}(Y-B_2)) = F^+(Y-\text{sInt } B_1) \cup F^-(Y-\text{sInt } B_2) = (X-F^-\text{sInt } B_1)) \cup (X-F^+(\text{sInt } B_2)) = X - [F^-(\text{sInt } B_1) \cap F^+(\text{sInt } B_2)]$ and thus $\text{sInt}[F^-(B_1) \cap F^+(B_2)] \supset F^-(\text{sInt } B_1) \cap F^+(\text{sInt } B_2)$

(6) => (7). Let $x \in X$, V_1 a semi-neighbourhood of F(x) and V_2 a semi-neighbourhood which intersects F(x), then there exists two semi-open sets U_1 and U_2 such that $U_1 \subset V_1$ and $U_2 \subset V_2$, $F(x) \subset U_1$ and $F(x) \cap U_2 \neq 0$, thus $x \in F^+(U_1) \cap F^-(U_2)$. By hypothesis $x \in F^+(U_1) \cap F^-(U_2) = F^+(\text{sInt } U_1) \cap F^-(\text{sInt } U_2) \subset \text{sInt}[F^+(U_1) \cap F^-(U_2)] \subset \text{sInt } [F^+(V_1) \cap F^-(V_2)] \subset F^+(V_1) \cap F^-(V_2)$. From $x \in \text{sInt}[F^+(U_1) \cap F^-(U_2)] \subset F^+(V_1) \cap F^-(V_2)$ it follows that $F^+(V_1) \cap F^-(V_2)$ is a semi-neighbourhood of x.

(7) => (8). Let $x \in X$, V_1 a semi-neighbourhood of F(x) and V_2 a semi-neighbourhood which intersects F(x), then $U = F^+(V_1) \cap F^-(V_2)$ is a semi-neighbourhood of x, $F(U) \subset V_1$ and $F(y) \cap V_2 \neq 0$, $\forall y \in U$.

(8) => (1). Evident.

COROLLARY 1. For a single valued mapping $f:X\to Y$ the following are equivalent:

1. f is irresolute at x.
2. For each semi-open set $G \subset Y$ with $f(x) \in G$, there results the relation

$$x \in Cl[Int\ f^{-}1(G)].$$

3. For any open set $U \subset X$ containing x and for any semi-open set $G \subset Y$ with $f(x) \in G$, there exists a non-empty open set $G_U \subset U$ such that $f(G_U) \subset G$.

COROLLARY 2. For a single valued mapping $f:X\to Y$ the following are equivalent:

1. f is irresolute.
2. $f^{-1}(G) \in SO(X)$, $\forall\ G \in SO(Y)$. (Definition 1.1 [4]).
3. For each semi-closed set $V \subset Y$, $f^{-1}(V)$ is a semi-closed set. (Theorem 1.4, [4]).
4. For each subset $B \subset Y$, $Int[Cl\ fY^{-1}(B)] \subset f^{-1}(Scl\ B)$.
5. For each subset $B \subset Y$, $Scl\ f^{-1}(B) \subset f^{-1}(Scl\ B)$. Theorem 1.6, [4])
6. For each subset $B \subset Y$, $sInt\ f^{-1}(B) \supset f^{-1}\ (sInt\ B)$.
7. For each point x of X and for each semi-neighbourhood V of f(x), $f^{-1}(V)$ is a semi-neighbourhood of x.
8. For each point x of X and for each semi-neighbourhood V of f(x) there is a semi-neighbourhood U of x such that $f(U) \subset V$.

3. CONTINUOUS MULTIFUNCTIONS AND IRRESOLUTE MULTIFUNCTIONS.

The notion of strongly continuous multifunctions was introduced in [9] as a generalization of the univocal strongly continuous mapping defined by Levine in [10].

DEFINITION 7. The multifunction $F:X\to Y$ is strongly lower semi-continuous (s.l.s.c.) if for each subset $B \subset Y$, $F^{-}(B)$ is a open set in X [9].

DEFINITION 8. The multifunction $F:X\to Y$ is strongly upper semi-continuous (s.u.s.c) if for each subset $B \subset Y$, $F^{+}(B)$ is an open set in X.

THEOREM 3. If $F:X\to Y$ is a multifunction so that:

1. F is upper irresolute.
2. F is strongly lower semi-continuous, then F is irresolute.

PROOF. Let G_1, $G_2 \in SO(Y)$. Let $x \in F^{+}(G_1)$. F being upper irresolute then there is a semi-open set U containing x and $F(U) \subset G_1$. Since U is semi-open in X, then by Theorem 1 of [6], $x \in U \subset Cl[Int\ U] \subset Cl[Int\ F^{+}(G_1)]$. As x is chosen arbitrarily in $F^{+}(G_1)$ there follows that $F^{+}(G_1) \subset Cl[Int\ F^{+}(G_1)]$ and thus $F^{+}(G_1)$ is a semi-open set in X by Theorem 1 of [1]. F being s.l.s.c. then $F^{-}(G_2)$ is an open set in X. Then $F^{+}(G_1) \cap F^{-}(G_2) \in SO(X)$ and by Theorem 2, implication (2) => (1). F is irresolute.

DEFINITION 9. A multifunction $F:X\to Y$ is said to be injective if for x_1, $x_2 \in X$, $x_1 \neq x_2$ we have $F(X_1) \cap F(x_2) = 0$.

A multifunction $F:X\to Y$ is said to be pre-semi-open if for any semi-open set $A \subset X$ the set F(A) is semi-open.

DEFINITION 10. A set A is called regular open if A=Int[Cl A].

THEOREM 5. Let Y be a regular space and F:X→Y S(Y) be a pre-semi-open and irresolute multifunction. If one of the conditions holds:

1. Int F(X) = 0 for every $x \in X$.
2. F is injective,

Then F is lower semi-continuous.

PROOF. In a topological space (Y,T) the intersections of two regular open sets forms a base for a topology T_S on Y, called the semi-regularization of T. If the Y is a regular space then $T=T_S$. The proof follows then by Remark 1 and by Theorems 7 and 10 from [5].

THEOREM 6. Let Y be a regular space or a space which has a basis composed of open-closed sets. If F:X→K(Y) is a pre-semi-open, irresolute and injective multifunction, then F is continuous.

PROOF. Follows from Remark 1, Theorems 7 and 11 of [5] and Remark 8 from [5].

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