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FUNDAMENTAL THEOREM OF WIENER CALCULUS

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ABSTRACT. In this paper we define and develop a theory of differentiation in Wiener space C[0,T]. We then proceed to establish a fundamental theorem of the integral calculus for C[0,T]. First of all, we show that the derivative of the indefinite Wiener integral exists and equals the integrand functional. Secondly, we show that certain functionals defined on C[0,T] are equal to the indefinite integral of their Wiener derivative.

KEY WORDS AND PHRASES. Wiener (measure, integral, derivative, absolute continuity), Lebesgue absolute continuity, fundamental theorem.

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1. INTRODUCTION.

Consider the Wiener measure space $(C[0,T], \mathscr{F}^*, m_w)$ where C[0,T] is the space of all continuous functions x on [0,T] vanishing at the origin. For each partition $\tau = \tau_n = \{t_1,\cdots,t_n\}$ of [0,T] with $0 = t_0 < t_1 < \cdots < t_n = T$, let $X_\tau: C[0,T] \to \mathbb{R}^n$ be defined by $X_\tau(x) \equiv x(\tau) = (x(t_1),\cdots,x(t_n))$. Let $\mathscr{B}^n$ be the σ–algebra of Borel sets in $\mathbb{R}^n$. Then a set of the type

$$I = \{x \in C[0,T] : X_\tau(x) \in B\} \equiv X_\tau^{-1}(B) \ , \ B \in \mathscr{B}^n$$

is called a Wiener interval (or a Borel cylinder). It is well known that

$$m_w(I) = \int_B K(\tau,\vec{\eta})d\vec{\eta} \ , \tag{1.1}$$

where

$$K(\tau,\vec{\eta}) = \left\{ \prod_{j=1}^{n} 2\pi(t_j-t_{j-1})\right\}^{-1/2} \exp\left\{-\frac{1}{2}\sum_{j=1}^{n} \frac{(\eta_j-\eta_{j-1})^2}{t_j-t_{j-1}}\right\} \tag{1.2}$$

with $\vec{\eta} = (\eta_1,\cdots,\eta_n)$, and $\eta_0 = 0$. The measure m_w is a probability measure defined on the algebra $\mathscr{I}$ of all Wiener intervals and m_w is extended to the Caratheodory extension $\mathscr{I}^*$ of $\mathscr{I}$. Let $\mathscr{I}_\tau$ be the σ-algebra generated by the set $\{X_\tau^{-1}(B) : B \in \mathscr{B}^n\}$ with τ fixed. Then, by the definition of conditional expectation, see Doob [1], Tucker [2] and Yeh [3], for each Wiener integrable function F(x),

$$\begin{aligned} \mu_\tau(B) \equiv \int_{X_\tau^{-1}(B)} F(x)m_w(dx) &= \int_{X_\tau^{-1}(B)} E(F|\mathscr{I}_\tau)m_w(dx) \\ &= \int_B E(F(x)|X_\tau(x) = \vec{\eta})P_{X_\tau}(d\vec{\eta}) \ , \ B \in \mathscr{B}^n, \end{aligned} \tag{1.3}$$

where $P_{X_\tau}(B) = m_w(X_\tau^{-1}(B))$, and $E(F(x)|X_\tau(x) = \vec{\eta})$ is a Lebesgue measurable function of $\vec{\eta}$ which is unique up to null sets in $\mathbb{R}^n$. Also, using (1.1) and (1.3) and choosing $F(x) \equiv 1$, we see that

$$P_{X_\tau}(d\vec{\eta}) = K(\tau,\vec{\eta})d\vec{\eta} \ , \tag{1.4}$$

or

$$\frac{dP_{X_\tau}}{d\vec{\eta}} = K(\tau,\vec{\eta}) \ , \ \vec{\eta} \in \mathbb{R}^n \ . \tag{1.5}$$

Next, for each $F \in L_1(C[0,T],m_w)$ and each partition τ of C[0,T], let

$$F_\tau = E(F|\mathscr{I}_\tau) \tag{1.6}$$

and

$$\tilde{F}(\vec{\eta}) = E(F(x)|X_\tau(x) = \vec{\eta}) \equiv E(F|X_\tau)(\vec{\eta}) \ . \tag{1.7}$$

Then, $\{F_\tau\}$ is a martingale, and by the martingale convergence theorem,

$$\lim_{\|\tau\|\to 0} F_\tau(x) = F(x) \tag{1.8}$$

for almost all $x \in C[0,T]$. Furthermore,

$$F(x) = \lim_{\|\tau\|\to 0} E(F(y)|X_\tau(y) = x(\tau)) = \lim_{\|\tau\|\to 0} \tilde{F}(x(\tau)) \tag{1.9}$$

for almost all $x \in C[0,T]$.

For a given partition $\tau = \tau_n$ of [0,T] and $x \in C[0,T]$, define the polygonal function $[x] \equiv [x(\tau)]$ on [0,T] by

$$[x](t) = x(t_{j-1}) + \frac{t-t_{j-1}}{t_j-t_{j-1}}(x(t_j) - x(t_{j-1})) \ , \ t_{j-1} \le t \le t_j \ , \ j = 1,\cdots,n.$$

Similarly, for each $\vec{\eta} = (\eta_1,\cdots,\eta_n) \in \mathbb{R}^n$, define the polygonal function $[\vec{\eta}]$ of $\vec{\eta}$ on $[0,T]$ by

$$[\vec{\eta}](t) = \eta_{j-1} + \frac{t-t_{j-1}}{t_j-t_{j-1}}(\eta_j - \eta_{j-1}) \ , \ t_{j-1} \le t \le t_j \ , \ j = 1,\cdots,n \text{ with } \eta_0 = 0.$$

Then both functions $[x]$ and $[\vec{\eta}]$ are continuous on $[0,T]$, their graphs are line segments on each subinterval $[t_{j-1},t_j]$, and $[x](t_j) = x(t_j)$ and $[\vec{\eta}](t_j) = \eta_j$ at each $t_j \in \tau$.

For $x,y \in C[0,T]$, we use the convention:

$$x \le y \text{ if and only if } x(t) \le y(t) \text{ for every } t \in [0,T] \ ,$$

and

$$x < y \text{ if and only if } x(t) < y(t) \text{ for every } t \in (0,T] \ .$$

The main purpose of this paper is to define and develop a theory of differentiation in Wiener space $C[0,T]$, and then to establish a fundamental theorem of the integral calculus on $C[0,T]$; namely, that the Wiener derivative of the indefinite integral $\int_{y\le x} F(y)m_w(dy)$ is $F(x)$, and that a Wiener absolutely continuous function can be expressed as the indefinite integral of its Wiener derivative. This study was initiated by Smolowitz [4]. In this paper we incorporate some recent results of Park and Skoug [5] to improve and substantially simplify the concepts and results of Smolowitz [4].

2. THE WIENER DERIVATIVE.

Our first objective is to define the Wiener derivative $\mathscr{D}_x(\cdot)$ so that

$$\mathscr{D}_x \int_{y\le x} F(y)m_w(dy) = F(x)$$

for $F \in L_1(C[0,T],m_w)$. We start by quoting the following theorem from Park and Skoug [5] which plays an important role in this paper.

THEOREM A. Let $F \in L_1(C[0,T],m_w)$. Then for any Borel set $B \in \mathscr{B}^n$,

$$\mu_\tau(B) \equiv \int_{X_\tau^{-1}(B)} F(x)m_w(dx) = \int_B E_x[F(x) - [x] + [\vec{\eta}]]P_{X_\tau}(d\vec{\eta}) \tag{2.1}$$

where

$$E_x[F(x - [x] + [\vec{\eta}])] = \int_{C[0,T]} F(x - [x] + [\vec{\eta}])m_w(dx).$$

In view of (1.3) and (2.1), we may conclude that

$$E(F(x)|X_\tau(x) = \vec{\eta}) = E_x[F(x - [x] + [\vec{\eta}])] \tag{2.2}$$

for almost all $\vec{\eta}$ in $\mathbb{R}^n$; i.e., we may express the conditional expectation $E(F|X_\tau)(\vec{\eta})$ in terms of an ordinary Wiener integral. Note that for $F \in L_1(C[0,T],m_w)$, $\tilde{F}(\vec{\eta}) \equiv E(F|X_\tau)(\vec{\eta})$ is in $L_1(\mathbb{R}^n,P_{X_\tau}(d\vec{\eta}))$. Also note that for each $x \in C[0,T]$ and each partition $\tau = \{t_1,\cdots,t_n\}$ of $[0,T]$, $\tilde{F}(x(\tau)) = E(F(y)|X_\tau(y) = x(\tau))$ is a function of $x(t_1),\cdots,x(t_n)$.

DEFINITION 1. Let $F \in L_1(C[0,T],m_w)$. For each partition $\tau = \{t_1,\cdots,t_n\}$ of $[0,T]$ define the operator $\mathscr{D}_{x(\tau)}$ by

$$\mathscr{D}_{x(\tau)}F(x) = \frac{\partial^n \tilde{F}(x(\tau))}{\partial x(t_n)\cdots\partial x(t_1)} \Big/ K(\tau,x(\tau)) \tag{2.3}$$

if it exists. Furthermore, if $\mathscr{D}_{x(\tau)}F(x)$ exists for each partition τ, then the Wiener derivative of $F(x)$ is defined by

$$\mathscr{D}_x F(x) = \lim_{\|\tau\|\to 0} \mathscr{D}_{x(\tau)}F(x)$$

if the limit exists.

Our first theorem is the first half of the fundamental theorem of Wiener calculus.

THEOREM 1. Let $F \in L_1(C[0,T],m_w)$. Then

$$\mathscr{D}_x \int_{y\le x} F(y)m_w(dy) = F(x)$$

for almost all $x \in C[0,T]$.

PROOF. For $x \in C[0,T]$ let $G(x)$ denote the indefinite Wiener integral

$$G(x) = \int_{y\le x} F(y)m_w(dy) = E_y[I_x(y)F(y)] \tag{2.4}$$

where $I_x(y)$ is the indicator function

$$I_x(y) = \begin{cases} 1 & , \quad y(t) \le x(t) \quad \text{for all} \quad t \in [0,T] \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Then using (1.7), (2.4), (2.2), (1.3), (2.2) and the Fubini theorem, we obtain

$$\begin{aligned} \tilde{G}(\vec{\eta}) &= E(G(u)|X_\tau(u) = \vec{\eta}) \\ &= E_u(E_y[I_u(y)F(y)]|X_\tau(u) = \vec{\eta}) \\ &= E_u[E_y[I_{u-[u]+[\vec{\eta}]}(y)F(y)]] \\ &= E_u\Big[\int_{\mathbb{R}^n} E_y(I_{u-[u]+[\vec{\eta}]}(y)F(y)|X_\tau(y) = \vec{\xi})P_{X_\tau}(d\vec{\xi})\Big] \\ &= \int_{\mathbb{R}^n} E_u[E_y[I_{u-[u]+[\vec{\eta}]}(y-[y]+[\vec{\xi}])F(y-[y]+[\vec{\xi}])]]P_{X_\tau}(d\vec{\xi}) . \end{aligned} \tag{2.5}$$

But $I_{u-[u]+[\vec{\eta}]}(y-[y]+[\vec{\xi}])$ is zero unless

$$y(t) - [y](t) + [\vec{\xi}](t) \le u(t) - [u](t) + [\vec{\eta}](t) \tag{2.6}$$

for all $t \in [0,T]$. But (2.6) implies that

$$\xi_j = y(t_j) - [y](t_j) + [\vec{\xi}](t_j) \le u(t_j) - [u](t_j) + [\vec{\eta}](t_j) = \eta_j$$

for $j = 1,\cdots,n$. Hence we can write

$$\tilde{G}(\vec{\eta}) = \int_{-\infty}^{\eta_n} \cdots \int_{-\infty}^{\eta_1} E_u[E_y[I_{u-[u]+[\vec{\eta}]}(y-[y]+[\vec{\xi}]) \cdot F(y-[y]+[\vec{\xi}])]]K(\tau,\vec{\xi})d\xi_1 \cdots d\xi_n,$$

and so for each $x \in C[0,T]$,

$$\tilde{G}(x(\tau)) = \int_{-\infty}^{x(t_n)} \cdots \int_{-\infty}^{x(t_1)} E_u[E_y[I_{u-[u]+[x(\tau)]}(y-[y]+[\vec{\xi}]) \cdot F(y-[y]+[\vec{\xi}])]]K(\tau,\vec{\xi})d\xi_1 \cdots d\xi_n.$$

Hence

$$\begin{aligned}\frac{\partial^n \tilde{G}(x(\tau))}{\partial x(t_n) \cdots \partial x(t_1)} &= E_u[E_y[I_{u-[u]+[x(\tau)]}(y-[y]+[x(t)]) \\ &\quad \cdot F(y-[y]+[x(\tau)])]]K(\tau,x(\tau)) \\ &= E_{u,y}(I_u(y)F(y) \mid X_\tau(y) = x(\tau),\ X_\tau(u) = x(\tau))K(\tau,x(\tau)).\end{aligned} \tag{2.7}$$

Applying (1.9) to (2.7) yields

$$\begin{aligned}\mathscr{D}_x(G(x)) &= \lim_{\|\tau\| \to 0} E_{u,y}(I_u(y)F(y) \mid X_\tau(y) = x(\tau), X_\tau(u) = x(\tau)) \\ &= F(x)\end{aligned} \tag{2.8}$$

for almost all x in $C[0,T]$ which concludes the proof of Theorem 1.

COROLLARY 1. If $\{t_1', \cdots, t_m'\} \subseteq \tau = \{t_1, \cdots, t_n\}$ and if $F(y) = f(y(t_1'), \cdots, y(t_m'))$ is in $L_1(C[0,T], m_w)$, then

$$\mathscr{D}_{x(\tau)} \int_{y \le x} F(y)m_w(dy) = F(x)E_{u,y}(I_u(y) \mid X_\tau(y) = X_\tau(u) = x(\tau))$$

and

$$\mathscr{D}_x \int_{y \le x} F(y)m_w(dy) = F(x) = f(x(t_1'), \cdots, x(t_n'))$$

for almost all x in $C[0,T]$.

PROOF. Using (2.7) and (2.4) we see that

$$\mathscr{D}_{x(\tau)} \int_{y \le x} F(y)m_w(dy) = E_{u,y}(I_u(y)F(y) \mid X_\tau(y) = x(\tau),\ X_\tau(u) = x(\tau)).$$

Under the conditioning $X_\tau(y) = x(\tau)$, $F(y)$ becomes $f(x(t_1'), \cdots, x(t_m'))$ which equals $F(x)$. Therefore,

$$\mathscr{D}_{x(\tau)} \int_{y \le x} F(y)m_w(dy) = F(x)E_{u,y}(I_u(y) \mid X_\tau(y) = X_\tau(u) = x(\tau)).$$

As $\|\tau\| \to 0$, $E_{u,y}(I_u(y) \mid X_\tau(y) = X_\tau(u) = x(\tau)) \to I_x(x) = 1$ by (1.9) for almost all x in $C[0,T]$. Thus Corollary 1 is established.

COROLLARY 2. Let $\tau' = \{t_1', \cdots, t_m'\}$ be any partition of $[0,T]$, and let $F(x) = f(x(t_1'), \cdots, x(t_m'))$ be in $L_1(C[0,T], m_w)$. Then $\mathscr{D}_x F(x) = 0$.

PROOF. Let τ be a partition of $[0,T]$ properly containing τ'. Then

$$\tilde{F}(x(\tau)) = E(F|y)|X_\tau(y) = x(\tau)) = f(x(t_1'),\cdots,x(t_n')).$$

Thus $\mathscr{D}_{x(\tau)}F(x) = 0$, and so $\mathscr{D}_x F(x) = 0$.

3. LEBESGUE AND WIENER ABSOLUTE CONTINUITY.

In this section we show that certain functions defined on $C[0,T]$ are equal to the indefinite integral of their Wiener derivative.

For $\vec{a} = (a_1,\cdots,a_n)$ and $\vec{b} = (b_1,\cdots,b_n)$ in $\mathbb{R}^n$ with $a_i < b_i$, $i = 1,\cdots,n$, let $V(\vec{a},\vec{b},k)$ be the collection of all points of the form $\vec{v} = (v_1,\cdots,v_n)$ where each v_i is either a_i or b_i and exactly k of the v_i are a_i's. For any function f defined on $V(\vec{a},\vec{b},k)$ for $k = 0,1,\cdots,n$, let

$$\Delta_{\vec{a},\vec{b}} f = f(\vec{b}) + \sum_{k=1}^{n} (-1)^k \sum_{\vec{v}\in V(\vec{a},\vec{b},k)} f(\vec{v}) . \tag{3.1}$$

A function of n variables $f(u_1,\cdots,u_n)$ is said to be Lebesgue absolutely continuous in the sense of Vitali (see Clarkson and Adams [6,7] and Hobson [8]) on the region $\Omega \subset \mathbb{R}^n$ if, given $\varepsilon > 0$, there is a $\delta > 0$ such that if $I_k^* \equiv \times_{i=1}^{n} (\alpha_i^{(k)},\beta_i^{(k)}]$, $k = 1,2,\cdots$, are disjoint n-dimensional rectangles contained in Ω with $\sum_{k=1}^{N} m_L(I_k^*) < \delta$ for any N, then $\sum_{k=1}^{N} |\Delta_{\vec{\alpha}^{(k)},\vec{\beta}^{(k)}} f| < \varepsilon$, where $m_L(\cdot)$ denotes n-dimensional Lebesgue measure, and $\vec{\alpha}^{(k)} = (\alpha_1^{(k)},\cdots,\alpha_n^{(k)})$. A function $f(u_1,\cdots,u_n)$ is said to be Lebesgue absolutely continuous (in the sense of Hardy–Krause; see Berkson and Gillespie [9], and Clarkson and Adams [6,7]) on a region $\Omega \subset \mathbb{R}^n$ if for each $k = 1,\cdots,n-1$, whenever n–k variables are fixed then f, as a function of its remaining k variables, is Lebesgue absolutely continuous in the sense of Vitali on $\Omega \cap \mathbb{R}^k$. When we merely state "Lebesgue absolutely continuous", it is always meant in the sense of Hardy–Krause.

It is well known that if $f(u_1,\cdots,u_n)$ is Lebesgue absolutely continuous in the sense of Vitali on $R \equiv \times_{i=1}^{n} [a_i,b_i]$, then $\partial^n f(u_1,\cdots,u_n)/\partial u_1\cdots\partial u_n$ exists a.e. on R and is integrable on R. Furthermore

$$\int_R [\partial^n f(u_1,\cdots,u_n)/\partial u_1\cdots\partial u_n] du_1\cdots du_n = \Delta_{\vec{a},\vec{b}} f, \tag{3.2}$$

and

$$\int_R |\partial^n f(u_1,\cdots,u_n)/\partial u_1\cdots\partial u_n| du_1\cdots du_n = \mathrm{Var}(f,R) \tag{3.3}$$

where $\mathrm{Var}(f,R)$ denotes the total variation of f over R.

Let $G(x)$ be any Wiener integrable function on $C[0,T]$. Then, by definition, $\tilde{G}(\vec{\eta}) = E(G|X_\tau)(\vec{\eta})$ is a function of $\vec{\eta}$ which is integrable with respect to $P_{X_\tau}(d\vec{\eta}) = K(\tau,\vec{\eta})d\vec{\eta}$.

DEFINITION 2. A Wiener integrable function $G(x)$ defined on $C[0,T]$ is said to be Wiener absolutely continuous provided that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if the sequence $I_k \equiv \{x \in C[0,T] : \alpha_i^{(k)} < x(s_i^{(k)}) \leq \beta_i^{(k)},\ i = 1,\cdots,m^{(k)}\}$ with $-\infty \leq \alpha_i^{(k)} < \beta_i^{(k)} \leq \infty$ are disjoint Wiener intervals with $\sum_{k=1}^{N} m_w(I_k) < \delta$ for any N, then $\sum_{k=1}^{N} |\Delta_{\vec{\alpha}^{(k)},\vec{\beta}^{(k)}}\tilde{G}| < \varepsilon$.

The following propositions can be easily established.

PROPOSITION A. If $G(x)$ is Wiener absolutely continuous on $C[0,T]$, then for every partition τ of $(0,T]$, $\tilde{G}(\vec{\eta})$ is Lebesgue absolutely continuous on $\mathbb{R}^{|\tau|}$, where $|\tau|$ denotes the number of points in τ.

PROPOSITION B. Let $F \in L_1(C[0,T],m_w)$. Then the indefinite Wiener integral $G(x) = \int_{y\leq x} F(y)m_w(dy)$ is Wiener absolutely continuous on $C[0,T]$.

Our next theorem is the second half of the fundamental theorem of Wiener Calculus.

THEOREM 2. Let $G \in L_1(C[0,T],m_w)$ satisfy the conditions:

(i) $\mathscr{D}_x G(x)$ exists for almost all $x \in C[0,T]$ and belongs to $L_1(C[0,T],m_w)$,

(ii) $G(x)$ is Wiener absolutely continuous on $C[0,T]$,

(iii) If $\{x_k\}_{k=1}^{\infty}$ is a sequence in $C[0,T]$ such that $x_k(s_0) \to -\infty$ as $k \to \infty$ for some fixed point $s_0 \in (0,T]$, then $G(x_k) \to 0$ as $k \to \infty$.

Then

$$G(u) = \int_{x\leq u} \mathscr{D}_x G(x)m_w(dx) \tag{3.4}$$

for almost all u in $C[0,T]$.

PROOF. For given $\varepsilon > 0$ let $\delta = \delta(\varepsilon/3) > 0$ be the value for the Wiener absolute continuity of $G(x)$, and also assume that

$$m_w(S) < \delta \Rightarrow \int_S |\mathscr{D}_x G(x)|m_w(dx) < \varepsilon/3.$$

Let $\{\tau^{(k)}\}$ be a sequence of partitions of $[0,T]$ such that $\|\tau^{(k)}\| \to 0$ as $k \to \infty$. Then

$$\lim_{k\to\infty} \mathscr{D}_{x(\tau^{(k)})} G(x) = \mathscr{D}_x G(x) \text{ for almost all } x \in C[0,T].$$

By Egoroff's theorem, there exists a set $C_\varepsilon \subset C[0,T]$ with $m_w(C_\varepsilon) > 1 - \delta/2$ and a positive integer k_0 such that if $k \geq k_0$, then

$$|\mathscr{D}_{x(\tau^{(k)})}G(x) - \mathscr{D}_x G(x)| < \varepsilon/3 \quad \text{for every} \quad x \in C_\varepsilon .$$

Let

$$C_k = \{x \in C[0,T] : |\mathscr{D}_{x(\tau^{(k)})}G(x) - \mathscr{D}_x G(x)| < \varepsilon/3\}, \ k \geq k_0 .$$

Then $C_\varepsilon \subset C_k$, and hence $m_w(C_k) \geq m_w(C_\varepsilon) > 1 - \delta/2$, and

$$\int_{C_k} |\mathscr{D}_{x(\tau^{(k)})}G(x) - \mathscr{D}_x G(x)| m_w(dx) < \varepsilon/3 \quad \text{for} \quad k \geq k_0 .$$

The complements satisfy $m_w(\tilde{C_k}) \leq m_w(\tilde{C_\varepsilon}) < \delta/2$ for $k \geq k_0$. Next consider fixed k, $k \geq k_0$ and let q denote the number of points in the partition $\tau^{(k)}$. Let

$$E_k \equiv \{\vec{\eta} = (\eta_1,\cdots,\eta_q) \in \mathbb{R}^q : \vec{\eta} = x(\tau^{(k)}) \text{ for some } x \in \tilde{C_k}\} .$$

Then,

$$m_w(\tilde{C_k}) = \int_{E_k} K(\tau^{(k)},\vec{\eta})d\vec{\eta} .$$

Since $K(\tau^{(k)},\vec{\eta})$ is bounded in $\vec{\eta}$ on $\mathbb{R}^q$ and $\int_{E_k} K(\tau^{(k)},\vec{\eta})d\vec{\eta} = m_w(\tilde{C_k}) < \delta/2$, we can find a countable sequence of disjoint q-dimensional rectangles $I_\ell^* = \times_{i=1}^{q} (\alpha_i^{(\ell)},\beta_i^{(\ell)}]$, $\ell = 1,2,\cdots$ such that $E_k \subset \bigcup_{\ell=1}^{\infty} I_\ell^*$, and

$$\int_{E_k} K(\tau^{(k)},\vec{\eta})d\vec{\eta} \leq \sum_{\ell=1}^{\infty} \int_{I_\ell^*} K(\tau^{(k)},\vec{\eta})d\vec{\eta} = \sum_{\ell=1}^{\infty} m_w(I_\ell) < \delta ,$$

where

$$I_\ell = \{x \in C[0,T] : \alpha_i^{(\ell)} < x(s_i^{(k)}) \leq \beta_i^{(\ell)} \text{ for each } s_i^{(k)} \in \tau^{(k)}\} .$$

Hence

$$\int_{\tilde{C_k}} |\mathscr{D}_{x(\tau^{(k)})}G(x)| m_w(dx) = \int_{E_k} |\partial^q \tilde{G}(\eta_1,\cdots,\eta_q)/\partial\eta_1\cdots\partial\eta_q| d\eta_1\cdots d\eta_q$$

$$\leq \sum_{\ell=1}^{\infty} \int_{\alpha_q^{(\ell)}}^{\beta_q^{(\ell)}} \cdots \int_{\alpha_1^{(\ell)}}^{\beta_1^{(\ell)}} |\partial^q \tilde{G}(\eta_1,\cdots,\eta_q)/\partial\eta_1\cdots\partial\eta_q| d\eta_1\cdots d\eta_q$$

$$= \sum_{\ell=1}^{\infty} \mathrm{Var}(\tilde{G},I_\ell^*) ,$$

where the last equality follows from (3.3). Now,

$$\mathrm{Var}(\tilde{G},I_\ell^*) = \sup \sum_i |\Delta_{\vec{\sigma}_i,\vec{\rho}_i} \tilde{G}(\vec{\eta})|$$

where the supremum is taken over all possible nets of I_ℓ^*, and each net has total Lebesgue measure equal to that of I_ℓ^*, and so the corresponding Wiener intervals have total measure equal to $m_w(I_\ell)$. Since $\sum_{\ell=1}^{\infty} m_w(I_\ell) < \delta(\varepsilon/3)$, by the Wiener absolute continuity of G, we have

$$\sum_{\ell=1}^{N} \sum_{i} |\Delta_{\vec{\sigma}_i,\vec{\rho}_i} \tilde{G}(\vec{\eta})| < \varepsilon/3 \text{ for every N and every net.}$$

Thus, by taking the supremum over all nets, we get

$$\sum_{\ell=1}^{N} \mathrm{Var}(\tilde{G},I_\ell^*) \le \varepsilon/3 \text{ for every N,}$$

and hence

$$\int_{\tilde{C}_k} |\mathscr{D}_{x(\tau^{(k)})} G(x)| m_w(dx) \le \varepsilon/3.$$

Thus, for every $k \ge k_0$,

$$\int_{C[0,T]} |\mathscr{D}_{x(\tau^{(k)})} G(x) - \mathscr{D}_x G(x)| m_w(dx)$$

$$\le \int_{C_k} |\mathscr{D}_{x(\tau^{(k)})} G(x) - \mathscr{D}_x G(x)| m_w(dx)$$

$$+ \int_{\tilde{C}_k} |\mathscr{D}_{x(\tau^{(k)})} G(x)| m_w(dx) + \int_{\tilde{C}_k} |\mathscr{D}_x G(x)| m_w(dx)$$

$$< \varepsilon .$$

In particular

$$\int_{[x]\le[u]} |\mathscr{D}_{x(\tau^{(k)})} G(x) - \mathscr{D}_x G(x)| m_w(dx) < \varepsilon \quad \text{for} \quad k \ge k_0,$$

where $[\cdot]$ corresponds to $\tau^{(k)}$. Hence

$$\lim_{k\to\infty} \left[\int_{[x]\le[u]} \mathscr{D}_{x(\tau^{(k)})} G(x) m_w(dx) - \int_{[x]\le[u]} \mathscr{D}_x G(x) m_w(dx) \right] = 0 .$$

Since $\{x \in C[0,T] : [x] \le [u]\} \to \{x \in C[0,T] : x \le u\}$ as $k \to \infty$, an application of the dominated convergence theorem yields

$$\lim_{k\to\infty} \int_{[x]\le[u]} \mathscr{D}_x G(x) m_w(dx) = \int_{x\le u} \mathscr{D}_x G(x) m_w(dx) .$$

Thus,

$$\lim_{k\to\infty} \int_{[x]\le[u]} \mathscr{D}_{x(\tau^{(k)})} G(x) m_w(dx) = \int_{x\le u} \mathscr{D}_x G(x) m_w(dx) . \tag{3.5}$$

On the other hand, using (2.3), (3.2) and (3.1), we see that for any $a \in C[0,T]$ with $a < u$,

$$\int_{[a]\leq[x]\leq[u]} \mathscr{D}_{x(\tau^{(k)})} G(x)m_w(dx)$$

$$= \int_{a(\tau^{(k)})\leq x(\tau^{(k)})\leq u(\tau^{(k)})} \mathscr{D}_{x(\tau^{(k)})} G(x)m_w(dx)$$

$$= \int_{a(s_q^{(k)})}^{u(s_q^{(k)})} \cdots \int_{a(s_1^{(k)})}^{u(s_1^{(k)})} [\partial^q \tilde{G}(\eta_1,\cdots,\eta_q)/\partial\eta_1\cdots\partial\eta_q]d\eta_1\cdots d\eta_q \tag{3.6}$$

$$= \Delta_{a(\tau^{(k)}),u(\tau^{(k)})} \tilde{G}$$

$$= \tilde{G}(u(\tau^{(k)})) + \sum_{\ell=1}^{q} (-1)^\ell \sum_{\vec{v}\in V(a(\tau^{(k)}),u(\tau^{(k)}),\ell)} \tilde{G}(\vec{v}) .$$

If we let $a(s_i^{(k)}) \to -\infty$ as $k \to \infty$ for $i = 1,\cdots,q$ in (3.6), then by assumption (iii), $\tilde{G}(\vec{v}) \to 0$ as $k \to \infty$ for every $\vec{v} \in V(a(\tau^{(k)}),u(\tau^{(k)}),\ell)$, $\ell \geq 1$. Thus (3.6) reduces to

$$\int_{[x]\leq[u]} \mathscr{D}_{x(\tau^{(k)})} G(x)m_w(dx) = \tilde{G}(u(\tau^{(k)})) . \tag{3.7}$$

In view of (1.9) and (3.5), we conclude that

$$\int_{x\leq u} \mathscr{D}_x G(x)m_w(dx) = G(u) \quad \text{for almost all } u \text{ in } C[0,T],$$

and so (3.4) is established.

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