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# BIHARMONIC EIGEN-VALUE PROBLEMS AND $L^p$ ESTIMATES

**CHAITAN P. GUPTA and YING C. KWONG**

Department of Mathematical Sciences
Northern Illinois University
DeKalb, IL 60115

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ABSTRACT. Biharmonic eigen-values arise in the study of static equilibrium of an elastic body which has been suitably secured at the boundary. This paper is concerned mainly with the existence of and $L^p$-estimates for the solutions of certain biharmonic boundary value problems which are related to the first eigen-values of the associated biharmonic operators. The methods used in this paper consist of the "a-priori" estimates due to Agmon-Douglas-Nirenberg and P. L. Lions along with the Fredholm theory for compact operators.

Key words: Biharmonic eigen-value problems, $L^p$-estimates, weak-solution, strong-solution, Fredholm theory, complementary boundary condition, boundary value problem.

AMS (MOS) Subject Classification: 35G15, 35J40, 35B45, 35B65.

## 1. INTRODUCTION.

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\Gamma$ and $\lambda \in \mathbb{R}$. Let $\Delta$ denote the Laplace operator on $\mathbb{R}^N$. Consider the following eigen-value problems for the biharmonic operator $\Delta^2$:

$$\begin{cases} \Delta^2 u = \lambda^2 u, & on\ \Omega, \\ u = 0, & on\ \Gamma, \\ \Delta u = 0, & on\ \Gamma; \end{cases} \tag{1.1}$$

and

$$\begin{cases} \Delta^2 u = \gamma^2 u, & on\ \Omega \\ \dfrac{\partial u}{\partial n} = 0, & on\ \Gamma, \\ \dfrac{\partial(\Delta u)}{\partial n} = 0, & on\ \Gamma; \end{cases} \tag{1.2}$$

and

$$\begin{cases} \Delta^2 u = \mu u, & on\ \Omega, \\ u = 0, & on\ \Gamma, \\ \dfrac{\partial u}{\partial n} = 0, & on\ \Gamma. \end{cases} \tag{1.3}$$

Here $\frac{\partial}{\partial n}$ denotes the exterior normal derivative at a point on the boundary $\Gamma$ of $\Omega$. The eigen-value problem (1.1) arises in the study of the static equilibrium of an elastic body $\overline{\Omega}$ which is simply supported along the boundary $\Gamma$ while problem (1.2) corresponds to the case when the boundary $\Gamma$ is supported by sliding clamps. The eigen-value problem (1.3) arises when the boundary $\Gamma$ of an elastic body $\overline{\Omega}$ is fixed or cantilevered.

The spectrum and the corresponding eigen-spaces for the problem (1.1) (respectively (1.2)) can be studied by considering the biharmonic operator associated to (1.1) (respectively (1.2)) as the squares of the second order Dirichlet (respectively Neumann) operator $-\Delta$ in $L^2(\Omega)$. Indeed, $\lambda^2$ is an eigen-value for (1.1) *iff* $\lambda$ is an eigen-value for the Dirichlet problem

$$\begin{cases} -\Delta u = \lambda u, & on\ \Omega \\ u = 0, & on\ \Gamma, \end{cases} \tag{1.1$^*$}$$

and $\gamma^2$ is an eigen-value for (1.2) *iff* $\gamma$ is an eigen-value for the Neumann problem

$$\begin{cases} -\Delta u = \gamma u, & on\ \Omega \\ \dfrac{\partial u}{\partial n} = 0, & on\ \Gamma, \end{cases} \tag{1.2$^*$}$$

(see [1]).

We note that the associated biharmonic operator in $L^2(\Omega)$ for the eigen-value problem (1.3) cannot be obtained as the square of any operator in $L^2(\Omega)$ defined in terms of the Laplacian operator $-\Delta$. Our approach then is to study the eigen-value problem (1.3) through the notion of weak-solutions and the Fredhlom theory of compact operators. Accordingly, we define in section 2, a biharmonic operation $\Delta^2$ associated to (1.3) which has $W_0^{2,2}(\Omega)$ as its domain in $L^2(\Omega)$.

The eigen-value problem (1.3) was studied earlier by other methods by Courant-Hilbert [2], Weinstein [3], Bazley, Fox and Stadter [4] and Fichera [5] (see also Weinstein-Stenger [6]).

The results concerning (1.3) in this paper (c.f. Theorem 2.2 (i), (ii) and (iii)) are about the general nature of the eigen-values and the eigen-spaces and are for general $\Omega$ in $\mathbf{R}^N$. The other works mentioned give methods for the computation or estimation of the eigen-values and the eigen-functions for (1.3) (refer to section 2 for more details). In general, they just deal with either a square or a circular domain in $\mathbf{R}^2$.

Furthermore, our approach for problem (1.3) also works for (1.1) and (1.2) through straightforward adaptations. Besides, our approach yields as a by-product, some preliminary $L^2$-estimates ((2.4), (3.6) and (3.7)) which are needed in the proofs of the main results of this paper.

The main interest of this paper is to obtain $L^p$-estimates on $u$, where $u$ is a solution of one of the following linear problems

$$\begin{cases} \Delta^2 u - \lambda_1^2 u = f, & on\ \ \Omega, \\ u = 0, & on\ \ \Gamma, \\ \Delta u = 0, & on\ \ \Gamma; \end{cases} \tag{1.4}$$

$$\begin{cases} \Delta^2 u = f, & on\ \ \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & on\ \ \Gamma, \\ \dfrac{\partial(\Delta u)}{\partial n} = 0, & on\ \ \Gamma; \end{cases} \tag{1.5}$$

$$\begin{cases} \Delta^2 u - \mu_1 u = f, & on\ \ \Omega, \\ u = 0, & on\ \ \Gamma, \\ \dfrac{\partial u}{\partial n} = 0, & on\ \ \Gamma. \end{cases} \tag{1.6}$$

Here $\lambda_1$ denotes the first positive eigen-value of the eigen-value problems (1.1), $\mu_1$ the first eigen-value of the eigen-value problem (1.3), and $f$ is in a certain subset of $L^p(\Omega)$, $2 \leq p < \infty$ (this will be specified later).

We call problems (1.1) and (1.4) the Biharmonic Dirichlet problems, (1.2) and (1.5) the Biharmonic Neumann problems, (1.3) and (1.6) the Mixed Biharmonic problems. The significance of these problems and their $L^p$ estimates lies on the consequence that they are fundamental to existence results for $4^{th}$ order non-linear elliptic problems which are treated by the authors in a forthcoming paper [7].

The strategy of this paper is to first obtain $L^2(\Omega)$ estimates for the weak solutions of (1.4), (1.5) and (1.6) through classical Fredholm theory for compact operators and then extrapolate to the $L^p$ estimates for higher $p$ by standard bootstrap techniques together with the use of estimates from Agmon-Douglis-Nirenberg [8] and P. L. Lions [9]. Consequently, the weak solutions are infact proved to be strong solutions in $W^{4,p}(\Omega)$ together with the $W^{4,p}$-estimates on $u$. In the rest of this paper, we will simply focus our attention on (1.3) and (1.6). The proofs in the cases of the other two types of problems are parallel to that of (1.3) and (1.6), we will therefore, omit the details and only mention the necessary modifications for the other two cases whenever it is necessary. This will be done in section 3.

## 2. THE BIHARMONIC PROBLEM.

DEFINITION 2.1. Let $D_3(\Delta^2) = W_0^{2,2}(\Omega)$. For a given $f \in L^2(\Omega)$, $u \in D_3(\Delta^2)$ is said to be a "weak solution" of the problem

$$\begin{cases} \Delta^2 u = f, & on \;\; \Omega, \\ u = 0, & on \;\; \Gamma, \\ \dfrac{\partial u}{\partial n} = 0, & on \;\; \Gamma, \end{cases} \tag{2.1}$$

if

$$\int_\Omega \Delta u \cdot \Delta \phi = \int_\Omega f \, \phi \tag{2.2}$$

for every $\phi \in D_3(\Delta^2)$.

$u \in W^{4,p}(\Omega)$, $p \geq 2$ is said to be a "strong solution" of (2.1) if $u$ satisfies (2.1) in the sense of trace. (Note that in this case, even the weak-solution satisfies the boundary conditions in the sense of trace since it belongs to $W_0^{2,2}(\Omega)$).

We observe that for $u$, $v \in D_3(\Delta^2)$,

$$(u,v)_{D_3} = \int_\Omega \Delta u \cdot \Delta v \tag{2.3}$$

defines an inner product on $D_3(\Delta^2)$ and we denote by $V$ the Hilbert space obtained by endowing $D_3(\Delta^2)$ with the norm induced by the inner product (2.3):

$$\|u\|_V = \int_\Omega |\Delta u|^2, \;\; u \in V. \tag{2.4}$$

Futhermore, $\| \; \|_V$ on $D_3(\Delta^2)$ is equivalent to the $W^{2,2}(\Omega)$ norm on $D_3(\Delta^2)$ in view of Theorem 1.1

We now get some preliminary $L^2$ estimates and spectrum results for (1.3) and (1.6).

THEOREM 2.1. Given $f \in L^2(\Omega)$, there exists exactly one $u \in V$ which is a weak solution of (2.1). Futhermore,

$$\|u\|_V \leq C(N,\Omega)\|f\|_{L^2(\Omega)} \tag{2.5}$$

and equivalently,

$$\|u\|_{W_0^{2,2}(\Omega)} \leq C(N,\Omega)\|f\|_{L^2(\Omega)} \tag{2.5}$$

where $C$ denotes different constants in (2.5) and (2.6) but both depend on $N$ and $\Omega$.

PROOF. Since $f \in L^2(\Omega)$ can be considered as an element of $W^{-2,2}(\Omega)$ whose action on $\phi \in W^{2,2}(\Omega)$ is defined by

$$f(\phi) = \int_\Omega f\,\phi, \tag{2.6}$$

and since the $W_0^{2,2}(\Omega)$ norm and the V-norm on $D_3(\Delta^2)$ are equivalent Hilbert space norms, we have by Riesz representation theorem that there exists exactly one $u \in V$ satisfying,

$$(u,\phi)_V = \int_\Omega \Delta u \cdot \Delta\phi = f(\phi) = \int_\Omega f\,\phi \tag{2.7}$$

for every $\phi \in V$ whence the existence and uniqueness results. Estimate (2.5) is now immediate from (2.7) and Theorem 1.1 of [9].

Hence the theorem.//

We now define a linear mapping $L_3 : D(L_3) \subset V \to L^2(\Omega)$ by setting

$$D(L_3) = \{u \in V : \exists f \in L^2(\Omega) \text{ such that } u \text{ is the weak solution of (2.1)}\}$$

and for $u \in D(L_3)$,

$$L_3 u = f. \tag{2.8}$$

Hence, by Theorem 2.1, we have for $u \in D(L_3)$,

$$\|u\|_V \leq C(N,\Omega)\|L_3 u\|_{L^2(\Omega)}, \tag{2.9}$$

or

$$\|u\|_{W_0^{2,2}(\Omega)} \leq C(N,\Omega)\|L_3 u\|_{L^2(\Omega)}. \tag{2.9}$$

The following theorem investigates the spectrum of the linear mapping $L_3 : D(L_3) \subset V \to L^2(\Omega)$ defined by (2.8).

THEOREM 2.2. The spectrum of $L_3$ is given by $\{0 < \mu_1 < \cdots < \mu_n < \cdots\}$ counted according to multiplicity and if $\{E_1, \ldots, E_n, \cdots\}$ are the corresponding eigen-spaces, then

(i) $\lim_{n\to\infty} \mu_n = +\infty$,

(ii) $\dim E_n < \infty$ for every $n$,

(iii) $\{E_1, \ldots E_n, \cdots\}$ forms a complete orthogonal system in $L^2(\Omega)$.

(iv) Furthermore, if we let

$$Lu = L_3 u - \mu_1 u,\ u \in D(L_3), \tag{2.10}$$

then

$$L^2(\Omega) = (\ker L) \oplus R(L), \tag{2.11}$$

where $R(L)$ denotes the range of $L$ in $L^2(\Omega)$ and $\oplus$ denotes the direct sum so that $R(L) = (\ker L)^\perp = E_1^\perp$ in (2.11).

(v) Furthermore, for any $f \in L^2(\Omega) \cap R(L)$, there exists an unique solution $u \in D(L_1) \cap R(L)$ such that $u$ satisfies

$$Lu = L_3 u - \mu_1 u = f$$

and hence $u$ is a weak solution of (1.6) in the sense that

$$\int_\Omega \Delta u \cdot \Delta\phi - \mu_1\int_\Omega u\,\phi = \int_\Omega f\,\phi,\ \forall\phi \in V. \tag{2.12}$$

Also we have the estimate

$$\|u\|_{L^2(\Omega)} \le \left(\frac{1}{\mu_2 - \mu_1}\right) \|f\|_{L^2(\Omega)} \tag{2.13}$$

PROOF. We first notice in view of Theorem 2.1 that, $L_3^{-1} : L^2(\Omega) \to W_0^{2,2}(\Omega)$ exists as a bounded linear mapping. We next assert that $L_3^{-1}$ is a positive-definite compact Hermitian operator on $L^2(\Omega)$.

To see $L_3^{-1}$ is Hermitian, it suffices to show that for $f, g \in D_3(\Delta^2)$

$$(L_3^{-1}f, g) = (f, L_3^{-1}g), \tag{2.14}$$

since $D_3(\Delta^2) = V$ is dense in $L^2(\Omega)$. Indeed, for $f, g \in D_3(\Delta^2)$, let $u = L_3^{-1}f$ and $v = L_3^{-1}g$, we have by the definition of $L_3$,

$$\int_\Omega \Delta u \cdot \Delta g = \int_\Omega f g = \int_\Omega \Delta v \cdot \Delta f$$

i.e.

$$(u, g)_V = (f, g)_{L^2(\Omega)} = (v, f)_V,$$

or

$$(L_3^{-1}f, g)_V = (f, g)_{L^2(\Omega)} = (f, L_3^{-1}g)_V. \tag{2.15}$$

Applying now the left hand equality of (2.15) to $f \in D_3(\Delta^2)$ and $L_3^{-1}g \in D_3(\Delta^2)$, we get

$$(L_3^{-1}f, L_3^{-1}g)_V = (f, L_3^{-1}g)_{L^2(\Omega)}. \tag{2.16}$$

Similarly, the right hand equality of (2.15) also gives

$$(L_3^{-1}f, g)_{L^2(\Omega)} = (L_3^{-1}f, L_3^{-1}g)_V \tag{2.17}$$

and (2.14) follows immediately from (2.16) and (2.17).

The positive definite property of $L_3^{-1}$ follows from the definition of $L_3$. Indeed, let $u = L_3^{-1}f$ for $f \in L^2(\Omega)$. Then,

$$(L_3^{-1}f, f)_{L^2(\Omega)} = (u, f)_{L^2(\Omega)} = \int_\Omega |\Delta u|^2 \ge 0$$

and

$$(L_3^{-1}f, f)_{L^2(\Omega)} = 0 \ \ iff \ \ u = 0 \ \ iff \ \ f = 0.$$

Finally, the compactness of $L_3^{-1} : L^2(\Omega) \to V \subset L^2(\Omega)$ follows immediately from the compact embedding $W^{2,2}(\Omega) \subset\subset\!\!\to L^2(\Omega)$. Invoking now the standard Fredholm theory for compact Hermitian operators, we see that if $\{\mu_1, \mu_2, \cdots, \mu_n, \cdots\}$ is the spectrum of $L_3$ with $\{E_1, \ldots, E_n, \ldots\}$ the corresponding eigen-spaces, we have

(i) $0 < \mu_1 < \mu_2 < \cdots < \mu_n < \cdots$ with $\lim_{n\to\infty} \mu_n = +\infty$,

(ii) $\dim E_n$ is finite for every $n$, and

(iii) the spaces $\{E_1, \ldots, E_n, \ldots\}$ form a complete orthogonal system of subspaces in $L^2(\Omega)$.

To prove (2.11), we see immediately, using Fredholm's alternative that $R(L) = (ker\ L)^{\perp}$ so that

$$L^2(\Omega) = (ker\ L) \oplus (ker\ L)^{\perp} = (ker\ L) \oplus R(L).$$

The uniqueness of the solution $u$ of (2.10) in $D(L_3) \cap R(L)$ is another simple consequence of the Fredholm theory. It now remains to prove estimate (2.13) (cf. Remark 2.1).

Now let $f \in L^2(\Omega) \cap R(L)$. There exists a unique solution $u$ in $D(L_3) \cap R(L)$ satisfying

$$Lu = L_3 u - \mu_1 u = f \tag{2.18}$$

i.e.

$$\begin{cases} \Delta^2 u - \mu_1 u = f, & on\ \Omega, \\ u = 0, & on\ \Gamma, \\ \dfrac{\partial u}{\partial n} = 0, & on\ \Gamma, \end{cases}$$

in the weak sense of (2.12). It follows that

$$\int_\Omega |\Delta u|^2 - \mu_1 \int_\Omega |u|^2 = \int_\Omega f u. \tag{2.19}$$

Also, since $u \in R(L) = (ker\ L)^{\perp} = E_1^{\perp}$, we have from the complete orthogonality of the sub-spaces $\{E_1, \ldots, E_n, \cdots\}$ and $0 < \mu_1 < \cdots < \mu_n < \cdots$ that

$$\int_\Omega |\Delta u|^2 \geq \mu_2 \int_\Omega |u|^2.$$

Invoking (2.19), we have

$$\begin{aligned} (\mu_2 - \mu_1)\int_\Omega |u|^2 &\leq \int_\Omega f u \leq \left(\int_\Omega |f|^2\right)^{1/2}\left(\int_\Omega |u|^2\right)^{1/2} \\ &\leq \frac{1}{2(\mu_2 - \mu_1)} \int_\Omega |f|^2 + \left(\frac{\mu_2 - \mu_1}{2}\right)\int_\Omega |u|^2, \end{aligned}$$

and consequently,

$$\int_\Omega |u|^2 \leq \frac{1}{(\mu_2 - \mu_1)^2}\int_\Omega |f|^2,$$

thus estimate (2.13) follows and the theorem has been proved.//

REMARK 2.1. We note that at this point, our solution $u$ in Theorem 2.2 is a weak solution in $W_0^{2,2}(\Omega)$; but in Theorem 2.4 with $f \in L^p(\Omega)$, $p \geq 2$, we will eventually show that $u$ is a strong solution in $W^{4,p}(\Omega)$. Let us remark that the spectrum of (1.6) is not related to the spectrums of (1.1)* and (1.2)* in any clear and precise manner as the spectrum of (1.1) is related to that of (1.1)* (respectively the spectrum of (1.2) to that of (1.2)*). Thus, it becomes an interesting topic to estimate the $\mu_1, \cdots, \mu_n, \cdots$ and to investigate their relationship with the spectrum of the other problems.

One of the most well known approach is the method of the intermediate problem (c.f. [6]) which consists of starting from a base problem (usually one with simplified boundary conditions) and then approaching the problem of interest through a sequence of intermediate problems (of which the spectrums are better known). As a result, one can obtain lower bounds on $\mu_1, \ldots, \mu_n, \cdots$. For instance, when $\Omega$ is a square, very good estimates have been obtained in [3] for the first four eigen-values:

$$13.294 \leq \mu_1 \leq 13.37,$$
$$50.41 \leq \mu_2 = \mu_3 \leq 55.76,$$
$$112.36 \leq \mu_4 \leq 134.56$$

Another approach is due to Fichera through the construction of intermediate Green's operators (c.f. [5]). This approach again leads to lower bounds on the $\mu_1, \ldots, \mu_n, \cdots$ for (1.3) when $\Omega$ is a square.

Finally, we have the decomposition method (c.f. [4]). In $R^2$, we can decompose (1.3) into $A^{[1]} + A^{[2]}$ where

$$\begin{cases} A^{[1]}u = u_{xxxx} + u_{yyyy}, & on \quad \Omega, \\ u = \dfrac{\partial u}{\partial n} = 0, & on \quad \partial\Omega, \\ A^{[2]}u = 2u_{xxyy}, & on \quad \Omega, \\ u = 0, & on \quad \partial\Omega \end{cases}$$

then both eigen-value problems for $A^{[1]}$ and $A^{[2]}$ are solved by separation of variables, say, when $\Omega$ is a rectangle.

The success of each of these approaches in getting explicit numerical estimates usually depends on the geometry of $\Omega$. A result which relates the spectrum of (1.3) to the spectrums of other problems in a most general setting is perhaps the following theorem. This result is obtained via the intermediate problem approach using (1.1)* as a base problem (for details refer to [3]).

THEOREM (2.2)*. If $\mu_1, \cdots, \mu_n, \cdots$ are the eigen-values of the vibrating clamped plate problem (1.3) and $\lambda_1, \ldots, \lambda_n, \cdots$ are the eigen-values of problem (1.1)*, then the two spectrums are related to each other by the inequality,

$$\lambda_n^2 < \mu_n.$$

It is not the main concern for this paper to investigate such estimates. The above remarks are included for the sake of completeness.

Now, we are ready to show that for $p \geq 2$, $f \in L^p(\Omega) \cap R(L)$, $u$ in Theorem 2.2 is in fact a strong solution and derive the corresponding $L^p$ estimates. But before we go further, we need to present a result from Agmon-Douglis-Nirenberg [8]. For reasons of simplicity, we shall only give a statement of this theorem as needed for the special case of this paper.

Consider the $4^{th}$ order elliptic problem

$$\begin{cases} \Delta^2 u + \beta u = f, \quad \beta \in \mathbb{R}, & on \quad \Omega, \\ B_1(x; \dfrac{\partial}{\partial x_1}, \ldots, \dfrac{\partial}{\partial x_N})u = 0, & on \quad \Gamma, \\ B_2(x; \dfrac{\partial}{\partial x_1}, \ldots, \dfrac{\partial}{\partial x_N})u = 0, & on \quad \Gamma, \end{cases} \tag{2.20}$$

where $B_1(x; \xi_1, \ldots, \xi_N)$ and $B_2(x; \xi_1, \ldots, \xi_N)$ are polynomials in $\xi_1, \ldots, \xi_N$, for $x \in \Gamma$, of order $m_1$ and $m_2$ respectively. Furthermore, let $\xi$ denote $(\xi_1, \ldots, \xi_N)$ and

$$P(\xi) = \xi_1^4 + \cdots + \xi_N^4 + 2\sum_{i>j} \xi_i^2 \xi_j^2 \tag{2.21}$$

be the corresponding characteristic polynomial of $\Delta^2$. We define the Agmon-Douglis-Nirenberg "Complementary Boundary Condition" as follows:

At any point $x \in \Gamma$, let $n(x)$ denote the normal to $\Gamma$ at $x$, and $\xi$ be any non-zero real vector parallel to the boundary at the point $x$. We require that the polynomials, in $\tau$, $B_j(x;\xi+\tau n(x))$, $j=1,2$ be linearly independent modulo the polynomial $(\tau-\tau_1^+(\xi))(\tau-\tau_2^+(\xi))$, where $\tau_1^+(\xi)$ and $\tau_2^+(\xi)$ are the roots of $P(\xi+\tau n(x))$ with positive imaginary parts (here $\tau$ is a scalar). Next, we need the following theorem from [8]:

THEOREM 2.3. Let $f\in L^p(\Omega)$, $p>1$, be given and let the complementary boundary condition be satisfied. Furthermore, let (2.20) has at most one solution. Then there exists a unique strong solution $u$ for (2.20) satisfying the estimate

$$\|u\|_{W^{4,p}(\Omega)}\le C(N,\Omega,\beta,p)\|f\|_{L^p(\Omega)} \tag{2.22}$$

REMARK 2.3. Theorem 2.3 says that uniqueness of solution is sufficient for solvability.

Equipped with Theorem 2.3, we are ready to prove the following theorem for $p\ge 2$.

THEOREM 2.4. Let $f\in L^p(\Omega)\cap E_1^\perp$ and $u$ be the unique weak solution of (1.6) in $D_3(\Delta^2)\cap E_1^\perp(E_1^\perp=R(L))$. Then $u$ is infact the strong solution of (1.6) in $W^{4,p}(\Omega)$ satisfying the estimate

$$\|u\|_{W^{4,p}(\Omega)}\le C(N,\Omega,p,\mu_1,\mu_2)\|f\|_{L^p(\Omega)}. \tag{2.23}$$

PROOF. To show that $u$ is a strong solution and to obtain higher $L^p$ estimate (2.23), we must fit the problem into the setting of Theorem 2.3. Therefore, we first note that the characteristic polynomials corresponding to $L_3u$, $B_1u=u$, $B_2u=\dfrac{\partial u}{\partial n}$ are respectively

$$\begin{aligned}
&P(\xi_1,\ldots,\xi_N)=\xi_1^4+\cdots+\xi_N^4+2\sum_{i>j}\xi_i^2\xi_j^2\\
&B_1(x;\xi_1,\ldots,\xi_N)\equiv 1\ \ (\textit{the constant polynomial } 1),\ x\in\Gamma\\
&B_2(x;\xi_1,\ldots,\xi_N)=n_1(x)\xi_1+\cdots+n_N(x)\xi_N,\ x\in\Gamma
\end{aligned} \tag{2.24}$$

where $n(x)=(n_1(x),\ldots,n_N(x))$ is the unit normal at $x\in\Gamma$. These polynomials satisfy the complementary boundary condition of Theorem 2.3 when $\Gamma$ is smooth. Consider now the elliptic problem

$$\begin{cases}\Delta^2 v=\mu_1 u+f, & on\ \ \Omega,\\ v=0, & on\ \ \Gamma,\\ \dfrac{\partial v}{\partial n}=0, & on\ \ \Gamma,\end{cases} \tag{2.25}$$

where $f\in L^p(\Omega)\cap E_1^\perp$, $p\ge 2$ and $u\in W_0^{2,2}(\Omega)$ are the same as that in Theorem 2.2. Obviously, $\mu_1u+f\in L^2(\Omega)$ and since $\beta=0$ is not an eigen-value of $L_3$, we have the uniqueness of solutions for (2.25). On invoking Theorem 2.3 with $\beta=0$, we get that there exists a unique strong solution $v$ for (2.25) in $L^2(\Omega)$. But $u$ being a weak solution of (1.4) must also satisfy (2.25) in the weak sense. We conclude by uniqueness of the weak solution in Theorem 2.1 that $u=v$ and hence $u$ is infact a strong solution in $W^{4,2}(\Omega)$ of (1.4) in view of Theorem 2.3.

Now we claim that $u$ is also in $L^p(\Omega)$. Indeed using the Sobolev embeddings,

$$\begin{aligned}
&(i)\ \ W^{4,q}(\Omega)\subset\subset\hookrightarrow C_B^{4-N/q}(\Omega) && \text{for } q>\frac{N}{4},\\
&(ii)\ \ W^{4,q}(\Omega)\subset\subset\hookrightarrow L^{\frac{Nq}{N-4q}}(\Omega) && \text{for } q<\frac{N}{4},\\
&(iii)\ \ W^{4,q}(\Omega)\subset\subset\hookrightarrow L^k(\Omega)\ \forall\, 1\le k<\infty && \text{for } q=\frac{N}{4},
\end{aligned} \tag{2.26}$$

we can start with $q = 2$ and argue inductively as follows:

Let $f \in L^p(\Omega)$ and $u \in L^q(\Omega)$, without loss of generality assume $p > q \geq 2$ and that

$$\begin{cases} \Delta^2 u = \mu_1 u + f, & on \ \Omega, \\ u = 0 & , \ on \ \Gamma, \\ \dfrac{\partial u}{\partial n} = 0 & , \ on \ \Gamma. \end{cases} \tag{2.27}$$

By the previous argument, $u \in W^{4,q}(\Omega)$. For $q \geq \frac{N}{4}$ we immediately have $u \in L^p(\Omega)$ due to the embeddings in (i) and (iii). If $q < \frac{N}{4}$, but $\frac{Nq}{N-4q} \geq p$, then $u \in L^p(\Omega)$ by embedding in (ii). Let us now assume that $q < \frac{N}{4}, \frac{Nq}{N-4q} < p$, then we have $\mu_1 u + f \in L^{\frac{Nq}{N-4q}}(\Omega)$. We conclude by using Theorem 2.3, as before, $u \in W^{4, \frac{Nq}{N-4q}}(\Omega)$. But

$$\frac{Nq}{N-4q} - q = \frac{4q^2}{N-4q} \geq \frac{16}{N-8} \text{ for } q \geq 2.$$

In view of this, after finitely many steps we must arrive at a stage at which $u \in L^k(\Omega)$ with $k \geq p$.

Since, now, $\mu_1 u + f \in L^p(\Omega)$, we have by invoking Theorem 2.3 with $\beta = 0$ that $u \in W^{4,p}(\Omega)$ and $u$ satisfies

$$\begin{aligned} \|u\|_{W^{4,p}(\Omega)} &\leq C(N,\Omega,p)\|\mu_1 u + f\|_{L^p(\Omega)} \\ &\leq C(N,\Omega,p,\mu_1)[\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}] \\ &\leq C(N,\Omega,p,\mu_1)[\|u\|_{L^2(\Omega)} + \|f\|_{L^p(\Omega)}] \end{aligned} \tag{2.28}$$

where we have used the well known norm inequality

$$\|u\|_{L^p(\Omega)} \leq \epsilon\|u\|_{W^{1,p}(\Omega)} + C(\epsilon,N,\Omega,p)\|u\|_{L^2(\Omega)} \tag{2.26}$$

for arbitrary $\epsilon > 0$ when $\Omega$ is bounded and $p \geq 2$. Finally, using the $L^2(\Omega)$ estimate (2.14) that we already established in Theorem 2.2, we obtain estimate (2.20) and the proof of the theorem is thus complete.//

## 3. $L^p$ - ESTIMATES ON THE BIHARMONIC DIRICHLET AND NEUMAN PROBLEM

As we mentioned in the introduction, the proofs of the $L^p$-estimates for these two types of problems are just parallel to that of the mixed Biharmonic problems. In this section, we will simply mention the necessary modifications and omit all the other repetitive details.

First of all, analogous to definition 2.1, we define the weak and strong solutions of

$$\begin{cases} \Delta^2 u = f, \ on \ \Omega \\ u = 0, \ on \ \Gamma, \\ \Delta u = 0, \ on \ \Gamma, \end{cases} \tag{3.1}$$

and that of

$$\begin{cases} \Delta^2 u = f, & on \quad \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & on \quad \Gamma, \\ \dfrac{\partial \Delta u}{\partial n} = 0, & on \quad \Gamma. \end{cases} \tag{3.2}$$

as follows.

DEFINITION 3.1. Let $D_1(\Delta^2) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, for a given $f \in L^2(\Omega)$, $u \in D_1(\Delta^2)$ is said to be a "weak-solution" of (3.1) if

$$\int_\Omega \Delta u \cdot \Delta\phi = \int_\Omega f\,\phi \tag{3.3}$$

for every $\phi \in D_1(\Delta^2)$.

$u \in W^{4,p}(\Omega)$, $p \geq 2$ is said to be a "strong-solution" of (3.1) if $u$ satisfies (3.1) in the sense of trace.

DEFINITION 3.2. Let $D_2(\Delta^2) = \{u \in W^{2,2}(\Omega) : \dfrac{\partial u}{\partial n} = 0 \ on \ \Gamma\}$, for a given $f \in L^2(\Omega)$, $u \in D_2(\Delta^2)$ is said to be a "weak solution" of (3.2) if

$$\int_\Omega \Delta u \cdot \Delta\phi = \int_\Omega f\,\phi \tag{3.4}$$

for every $\phi \in D_2(\Delta^2)$.

$u \in W^{4,p}(\Omega)$ is a "strong solution" of (3.2) if it satisfies (3.2) (or (1.5)) in the sense of trace.

Next, corresponding to the existence result and the $L^2$-estimate in part (v) of theorem 2.2, we have the following analogous theorems.

THEOREM 3.1. Let $E_1$ be the eigen-space of the first eigen-value $\lambda_1$ of (1.1)*, for any $f \in L^2(\Omega) \cap E_1^\perp$, there exists an unique weak solution $u \in D_1(\Delta^2) \cap E_1^\perp$ such that $u$ satisfies (1.4) in the sense

$$\int_\Omega \Delta u \cdot \Delta\phi - \lambda_1^2 \int_\Omega u\,\phi = \int_\Omega f\,\phi \quad \forall \phi \in D_1(\Delta^2) \tag{3.5}$$

Also, we have the estimate

$$\|u\|_{L^2(\Omega)} \leq \left(\frac{1}{\lambda_2^2 - \lambda_1^2}\right) \|f\|_{L^2(\Omega)} \tag{3.6}$$

where $\lambda_2$ is the second eigen-value of (1.1)*.

THEOREM 3.2. Let $E_0$ be the eigen-space of the first eigen-value $\gamma_0 = 0$ of (1.2)*, for any $f \in L^2(\Omega) \cap E_0^\perp$, there exists an unique weak solution $u \in D_2(\Delta^2) \cap E_0^\perp$ such that $u$ satisfies (1.5) in the sense of (3.4).

Also, we have the estimate

$$\|u\|_{L^2(\Omega)} \leq \frac{1}{\gamma_1^2} \|f\|_{L^2(\Omega)} \tag{3.7}$$

where $\gamma_1$ is the first non-zero eigen-value of (1.2)*.

Finally, we note that for (1.4), the characteristic polynomials corresponding to $L_1 u = \Delta^2 u$, $B_1 u = u$, $B_2 u = \Delta u$ are respectively

$$\begin{cases} L_1(\xi_1, \ldots, \xi_N) = \xi_1^4 + \cdots + \xi_N^4 + 2\sum_{i>j}\xi_i^2\xi_j^2, \\ B_1(x; \xi 1, \ldots, \xi_N) \equiv 1, \ x \in \Gamma, \\ B_2(x; \xi_1, \ldots, \xi_N) = \xi_1^2 + \cdots + \xi_N^2, \ x \in \Gamma, \end{cases} \tag{3.8}$$

and for (1.5), the characteristic polynomials corresponding to $L_2 u = \Delta^2 u$, $B_1 u = \frac{\partial u}{\partial n}$, $B_2 u = \frac{\partial(\Delta u)}{\partial n}$ are respectively

$$\begin{cases} L_2(\xi_1, \ldots, \xi_N) = \xi_2^4 + \cdots + \xi_N^4 + 2\sum_{i>j}\xi_i^2\xi_j^2, \\ B_1(x; \xi_1, \ldots, \xi_N) = n_1(x)\xi_1 + \cdots + n_N(x)\xi_N, \ x \in \Gamma, \\ B_2(x; \xi_1, \ldots, \xi_N) = (n_1(x)\xi_1 + \cdots + n_N(x)\xi_N)(\xi_1^2 + \cdots + \xi_N^2), \ x \in \Gamma. \end{cases} \tag{3.9}$$

It is trivial to see that both (3.8) and (3.9) satisfy the complementary boundary condition of Theorem 2.3.

Now we can proceed in exactly the same way as in the case of the mixed Biharmonic problem to get the following theorems which are the analogues of theorem 2.4 for the other two types of problems.

THEOREM 3.3. Let $f \in L^p(\Omega) \cap E_1^\perp$ for $p \geq 2$, and let $u$ be the unique weak solution of (1.4) in $D_1(\Delta^2) \cap E_1^\perp$ (as in Theorem 3.1), then $u$ is infact the strong unique solution of (1.4) in $W^{4,p}(\Omega)$ satisfying the estimate

$$\|u\|_{W^{4,p}(\Omega)} \leq C(N, \Omega, p, \lambda_1, \lambda_2)\|f\|_{L^p(\Omega)}. \tag{3.10}$$

THEOREM 3.4. Let $f \in L^p(\Omega) \cap E_0^\perp$ for $p \geq 2$, and let $u$ be the unique weak-solution of (1.5) in $D_2(\Delta^2) \cap E_0^\perp$ (as in Theorem 3.2), then $u$ is in fact the strong unique solution of (1.5) in $W^{4,p}(\Omega)$ satisfying the estimate

$$\|u\|_{W^{4,p}(\Omega)} \leq C(N, \Omega, p, \gamma_1)\ \|f\|_{L^p(\Omega)}. \tag{3.11}$$

REMARK 4.1. If $u$ in Theorem 2.4 belonged to $D_3(\Delta^2)$ instead of $D_3(\Delta^2) \cap E_1^\perp$, then estimate (2.23) may no longer hold. However, it is easy to see (by tracing back to the original proof of Theorem 2.4) that $u$ is still a strong solution and it satisfies the following weaker estimate

$$\begin{aligned} \|u\|_{W^{4,p}(\Omega)} &\leq C(N, \Omega, p, \mu_1)\ [\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}] \\ &\leq C(N, \Omega, p, \mu_1)[\|u\|_{L^2(\Omega)} + \|f\|_{L^p(\Omega)}]. \end{aligned} \tag{3.12}$$

This is due to the fact that we no longer have estimate (2.13) and $u$ is not unique in this case. Similar remarks apply to Theorem 3.3 and 3.4.

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