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*-TOPOLOGICAL PROPERTIES

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ABSTRACT. An *ideal* on a set X is a nonempty collection of subsets of X closed under the operations of subset (heredity) and finite unions (additivity). Given a topological space (X,τ) an ideal $\mathfrak{I}$ on X and $A\subseteq X$, $\psi(A)$ is defined as $\cup\{U\in\tau: U-A\in\mathfrak{I}\}$. A topology, denoted τ^*, finer than τ is generated by the basis $\{U-I: U\in\tau,\ I\in\mathfrak{I}\}$, and a topology, denoted $<\psi(\tau)>$, coarser than τ is generated by the basis $\psi(\tau)=\{\psi(U): U\in\tau\}$. The notation $(X,\tau,\mathfrak{I})$ denotes a topological space (X,τ) with an ideal $\mathfrak{I}$ on X. A bijection $f:(X,\tau,\mathfrak{I}) \rightarrow (Y,\sigma,\mathfrak{J})$ is called a *-homeomorphism if $f:(X,\tau^*) \rightarrow (Y,\sigma^*)$ is a homeomorphism, and is called a ψ-homeomorphism if $f:(X,<\psi(\tau)>) \rightarrow (Y,<\psi(\sigma)>)$ is a homeomorphism. Properties preserved by *-homeomorphisms are studied as well as necessary and sufficient conditons for a ψ-homeomorphism to be a *-homeomorphism. The semi-homeomorphisms and semi-topological properties of Crossley and Hildebrand [Fund. Math., LXXIV (1972), 233-254] are shown to be a special case.

KEYWORDS AND PHRASES. Ideal, regular open, semi-open, semi-homeomorphism, semi-topological property, semiregular, compatible ideal, topological property, *-topological property, τ-boundary ideal, nowhere dense sets, meager sets.

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1. INTRODUCTION

Given a topological space (X,τ), a nonempty collection of subsets $\mathfrak{I}$ on X is called an *ideal* [11] if the following hold:

I. If $A\in\mathfrak{I}$ and $B\subseteq A$, then $B\in\mathfrak{I}$ (heredity); and

II. If $A\in\mathfrak{I}$ and $B\in\mathfrak{I}$, then $A\cup B\in\mathfrak{I}$ (finite additivity).

An ideal is called a σ-*ideal* if the following holds:

III. If $\{A_n: n=1,2,3,...\}$ is a countable subcollection of $\mathfrak{I}$, then $\cup\{A_n: n=1,2,3,...\}\in\mathfrak{I}$ (countable additivity).

The notation $(X,\tau,\mathfrak{I})$ denotes a nonempty set X, a topology τ on X, and an ideal $\mathfrak{I}$ on X. Given a point $x\in X$, we denote by $\tau(x)$ the "τ neighborhood system at x"; i.e., $\tau(x)=\{U\in\tau: x\in U\}$.

Given a space $(X,\tau,\mathfrak{I})$ and a subset A of X, we denote by $A^*(\mathfrak{I},\tau)=\{x\in X: U\cap A\notin\mathfrak{I}$ for every $U\in\tau(x)\}$, the *local function of* A *with respect to* $\mathfrak{I}$ *and* τ [21]. When no ambiguity is present, we simply write A^* for $A^*(\mathfrak{I},\tau)$. We let $Cl^*(A)=A\cup A^*$ which defines a Kuratowski closure operator for a topology $\tau^*(\mathfrak{I})$ finer than τ (i.e. $\tau\subseteq\tau^*(\mathfrak{I})$), also denoted simply as τ^* when no ambiguity is present. A basis $\beta(\mathfrak{I},\tau)$ for τ^* can be described as follows [22]:

$\beta(\mathfrak{I},\tau)=\{U-I: U\in\tau,\ I\in\mathfrak{I}\}$. We will denote $\beta(\mathfrak{I},\tau)$ simply by β when no ambiguity is present.

Given spaces $(X,\tau,\mathfrak{I})$, $(Y,\sigma,\mathfrak{J})$, and a function $f:(X,\tau,\mathfrak{I}) \to (Y,\sigma,\mathfrak{J})$, we will call f a **-homeomorphism with respect to $\tau,\mathfrak{I},\sigma$, and $\mathfrak{J}$* if $f:(X,\tau^*) \to (Y,\sigma^*)$ is a homeomorphism, or simply a **-homeomorphism* when no ambiguity is present. A topological property P will be called a **-topological property with respect to $\tau,\mathfrak{I},\sigma$, and $\mathfrak{J}$* if it is preserved by any *-homeomorphism with respect to $\tau,\mathfrak{I},\sigma$ and $\mathfrak{J}$ or, following our convention, simply a **-topological property* when no ambiguity is present.

In this paper we study *-topological properties and show that the semi-topological properties of Crossley and Hildebrand [4] are a special case.

Given a space $(X,\tau,\mathfrak{I})$ and $A\subseteq X$, we denote by Int(A) and Cl(A) the interior and closure of A with respect to τ respectively, and by $\mathrm{Int}^*(A)$ and $\mathrm{Cl}^*(A)$ the interior and closure of A with respect to τ^* respectively. We abbreviate "if and only if" by "iff", and use the symbol "$\to$" to mean "implies" or "which implies" as fits the context. Of course the symbol "$\to$" is also used to denote functional correspondence; i.e. "$f:A\to B$".

2. *-HOMEOMORPHISMS AND SEMIREGULAR PROPERTIES

Since *-topological properties are defined as those preserved by *-homeomorphisms, sufficient conditions for a function to be a *-homeomorphism are a central issue.

DEFINITION. A space $(X,\tau,\mathfrak{I})$ is said to be *$\mathfrak{I}$-compact* [14, 19] if for every open cover $\{U_\alpha:\alpha\in\Delta\}$ of X, there exists a finite subcollection $\{U_{\alpha_i}:i=1,2,...,n\}$ such that $X-\cup\{U_{\alpha_i}:i=1,2,...,n\}\in\mathfrak{I}$.

Observe that whenever $\mathfrak{I}$ is an ideal on X and $f:X\to Y$ is a function, then $f(\mathfrak{I}) = \{f(I):I\in\mathfrak{I}\}$ is an ideal on Y.

The following theorem gives sufficient conditions for a function to be a *-homeomorphism.

THEOREM 2.1. [6] Let $f:(X,\tau,\mathfrak{I})\to(Y,\sigma)$ be a bijection with (X,τ) $\mathfrak{I}$-compact and (Y,σ) Hausdorff. If $f:(X,\tau^*) \to (Y,\sigma)$ is continuous, then f is a *-homeomorphism with respect to $\tau,\mathfrak{I},\sigma$, and $f(\mathfrak{I})$.

The following theorem shows that $\mathfrak{I}$-compactness is a *-topological property with respect to $\mathfrak{I}$ and $f(\mathfrak{I})$.

THEOREM 2.2. Let $f:(X,\tau,\mathfrak{I}) \to (Y,\sigma,f(\mathfrak{I}))$ be a *-homeomorphism. Then (X,τ) is $\mathfrak{I}$-compact iff (Y,σ) is $f(\mathfrak{I})$-compact.

PROOF. NECESSITY. Assume (X,τ) is $\mathfrak{I}$-compact and let $\{V_\alpha:\alpha\in\Delta\}$ be a σ-open cover of Y. Then $\{f^{-1}(V_\alpha):\alpha\in\Delta\}$ is a τ^*-open cover of X. It is shown in [14] that (X,τ) is $\mathfrak{I}$-compact iff (X,τ^*) is $\mathfrak{I}$-compact. Thus there exists a finite subcollection $\{f^{-1}(V_{\alpha_i}):i=1,2,...,n\}$ such that $X-\cup f^{-1}(V_{\alpha_i}) = I\in\mathfrak{I}$. Consequently, $Y-\cup V_{\alpha_i} = f(I)\in f(\mathfrak{I})$ and it is shown that (Y,σ) is $f(\mathfrak{I})$-compact.

SUFFICIENCY. Assume (Y,σ) is $f(\mathfrak{I})$-compact and let $\{U_\alpha:\alpha\in\Delta\}$ be a τ-open cover of X. Then $\{f(U_\alpha):\alpha\in\Delta\}$ ia a σ^*-open cover of Y, and there exists a finite subcollection $\{f(U_{\alpha_i}):i=1,2,...,n\}$ such that $Y-\cup f(U_{\alpha_i}) = f(I)\in f(\mathfrak{I})$. Then $X-\cup U_{\alpha_i} = I\in\mathfrak{I}$, and the proof is complete.

Given a space (X,τ), recall that a subset U of X is said to be *regular open* if $U = \mathrm{Int}(\mathrm{Cl}(U))$. We denote by $RO(X,\tau)$ the collection of all regular open subsets of (X,τ). The collection $RO(X,\tau)$ is a basis for a topology coarser than τ, denoted τ_s, called the *semiregularization of τ*.

DEFINITION. Given a function $f:(X,\tau) \to (Y,\sigma)$ from a space (X,τ) to a space (Y,σ), f is said to be a *δ-homeomorphism* [15] iff $f:(X,\tau_s) \to (Y,\sigma_s)$ is a homeomorphism. A topological property P will be called a *semiregular topological property* iff it is preserved by δ-homeomorphisms.

Another approach to semiregular topological properties is to define them to be topological properties shared by topologies which have the same semiregularizations. This approach is easily seen to be equivalent to the approach in this paper.

Given a space $(X,\tau,\mathfrak{I})$, $\mathfrak{I}$ is said to be *τ-boundary* [14] if $\tau\cap\mathfrak{I} = \{\emptyset\}$.

THEOREM 2.3 [9, Theorem 6.4]. Let $(X,\tau,\mathfrak{I})$ be a space with $\tau\cap\mathfrak{I} = \{\emptyset\}$, then $\tau_s = (\tau^*)_s$.

Using the previous theorem, we can show the following.

THEOREM 2.4. Let $f:(X,\tau,\mathfrak{I}) \to (Y,\sigma,\mathfrak{J})$ be a *-homeomorphism with $\tau\cap\mathfrak{I} = \{\emptyset\}$ and $\sigma\cap\mathfrak{J} = \{\emptyset\}$. Then any semi-regular property is a *-topological property.

PROOF. Let P be a semi-regular property and assume (X,τ) is P. Then (X,τ_s) is P by definition $\to(X,(\tau^*)_s)$ is P by Theorem 2.3 $\to$ (X,τ^*) is P by definition $\to$ (Y,σ^*) is P since semi-regular properties are topological $\to$ $(Y,(\sigma^*)_s)$ is P by definition $\to(Y,\sigma_s)$ is P by Theorem 2.3 $\to(Y,\sigma)$ is P by definition.

We denote by $\mathcal{N}(\tau)$ the ideal of nowhere dense sets with respect to τ. Given spaces (X,τ) and (Y,σ), *-topological properties with respect to $\tau, \mathcal{N}(\tau)$, σ, and $\mathcal{N}(\sigma)$ will be called *α-topological properties* since $\tau^*(\mathcal{N}(\tau))$ is commonly known as the α-topology in the literature, and is denoted τ^α [16,17].

Another approach to α-topological properties is to define them to be topological properties shared by τ and τ^α. This approach is easily seen to be equivalent to the one taken in this paper.

As an easy corollary to the previous theorem, we obtain the following result of Jankovic' and Reilly.

COROLLARY 2.5 [10] Semiregular properties are α-topological properties.

PROOF. Given a space (X,τ), observe that $\mathcal{N}(\tau)\cap\tau = \{\emptyset\}$ and apply Theorem 2.4.

The list of semiregular properties which have been established in the literature is quite extensive and includes: Hausdorff, Urysohn, almost regular, connected, extremally disconnected, H-closed, S-closed, light compact, and pseudocompact.

3. THE LIFTING THEOREM

Given a space (X,τ), a subset A of X is said to be *semi-open* [12] if there exists a $U\in\tau$ such that $U\subseteq A\subseteq Cl(U)$ or, equivalently, if $A\subseteq Cl(Int(A))$. A function $f:(X,\tau) \to (Y,\sigma)$ is said to be *pre-semi-open* [1] if for every semi-open set $A\subseteq X$, f(A) is semi-open in Y; and is said to be *irresolute* [4] if for every semi-open set $B\subseteq Y$, $f^{-1}(B)$ is semi-open in X. A bijection $f:(X,\tau) \to (Y,\sigma)$ is said to be a *semi-homeomorphism* [4] if it is both pre-semi-open and irresolute. Properties preserved by semi-homeomorphisms are said to be *semi-topological properties* [4].

It can be shown that semi-topological properties are α-topological properties as a consequence of Theorem 2.6 of [4] and Theorem 2 of [3]. We will establish this fact (specifically we will show that the semi-topological properties are precisely the α-topological properties) as a corollary to the Lifting Theorem proven in this section. First however we need several preliminary results.

In [13] Natkaniec defines an operator $\psi(\mathcal{I},\tau):\mathcal{P}(X) \to \tau$, where $(X,\tau,\mathcal{I})$ is a space and $\mathcal{P}(X)$ denotes the power set of X, as follows: for every $A\subseteq X$, $\psi(\mathcal{I},\tau)(A) = \{x$: there exists a $U\in\tau(x)$ such that $U-A\in\mathcal{I}\}$ and observes that $\psi(\mathcal{I},\tau)(A) = X-(X-A)^*$. We denote $\psi(\mathcal{I},\tau)$ simply by ψ when no ambiguity is present. The operator ψ has been studied in [7] where the following is observed:

$$\psi(A) = \cup\{U\in\tau:U-A\in\mathcal{I}\}.$$

Note that $\psi(A)$ is open for every $A\subseteq X$.

THEOREM 3.1. Given a space $(X,\tau,\mathcal{I})$,

$$\tau^*(\mathcal{I}) = \{A\subseteq X:A\subseteq\psi(A)\}.$$

PROOF. Denote $\{A\subseteq X:A\subseteq\psi(A)\}$ by σ. First, we show that σ is a topology. Observe that $\emptyset\subseteq\psi(\emptyset)$ and $X\subseteq\psi(X) = X$. Now if $A,B\in\sigma$, $A\cap B\subseteq\psi(A)\cap\psi(B) = \psi(A\cap B) \to A\cap B\in\sigma$. If $\{A_\alpha:\alpha\in\Delta\}\subseteq\sigma$, then $A_\alpha\subseteq\psi(\cup A_\alpha)$ for every $\alpha \to \cup A_\alpha\subseteq\psi(\cup A_\alpha)$, and we have shown that σ is a topology.

Now if $U\in\tau^*$, and $x\in U$, there exists a $V\in\tau(x)$ and $I\in\mathcal{I}$ such that $x\in V-I\subseteq U$. Clearly $V-U\subseteq I$ so that $V-U\in\mathcal{I}$by heredity, and hence $x\in\psi(U)$. Thus $U\subseteq\psi(U)$ and we have shown $\tau^*\subseteq\sigma$.

Now let $A\in\sigma$. We have by definition that $A\subseteq\psi(A) \to A\subseteq X-(X-A)^* \to (X-A)^*\subseteq X-A \to X-A$ is τ^*-closed and hence $A\in\tau^*$. Thus $\sigma = \tau^*$ and the proof is complete.

It is interesting to observe the the specific form of $A^*(\mathcal{N}(\tau),\tau) = Cl(Int(Cl(A)))$ for $A\subseteq X$ [21], and, consequently, in this case we have $\psi(A) = Int(Cl(Int(A)))$. It is known that $A^*(\mathcal{M}(\tau),\tau)$ is regular closed [21], where $\mathcal{M}(\tau)$ denotes the ideal of meager sets, and hence $\psi(A)$ is regular open in this case.

Given a space $(X,\tau,\mathcal{I})$, $\mathcal{I}$ is said to be *compatible with* τ [17], denoted $\mathcal{I}\sim\tau$, if the following holds for every $A\subseteq X$: if for every $x\in A$ there exists a $U\in\tau(x)$ such that $U\cap A\in\mathcal{I}$, then $A\in\mathcal{I}$. Ideals having this property are called "supercompact" in [21], "adherence ideals" in [22], and are said to have the "strong Banach's localization property" in [20]. For several characterizations of compatibility, see [9]. One significant consequence of $\mathcal{I}\sim\tau$ is that $\beta = \tau^*$ and all open sets in τ^* are of the *simple form* U-I where $U\in\tau$, $I\in\mathcal{I}$. However, we can have $\beta = \tau^*$ and $\mathcal{I}$ not be compatible with τ as the ideal of finite sets in an infinite discrete space shows.

It is known that $\mathcal{N}(\tau) \sim \tau$, and $\mathcal{M}(\tau) \sim \tau$ (this is known as the Banach Category Theorem) ([21], [18]). It is also known that if (X,τ) is hereditarily Lindelöf and $\mathfrak{I}$ is a σ-ideal then $\mathfrak{I} \sim \tau$ [9].

A convenient characterization of $\mathfrak{I} \sim \tau$ is the following.

THEOREM 3.2. [7] Let $(X,\tau,\mathfrak{I})$ be a space. Then $\mathfrak{I} \sim \tau$ iff $\psi(A)\text{-}A \in \mathfrak{I}$ for every $A \subseteq X$.

The following result is a straightforward consequence of the previous theorem.

THEOREM 3.3. Let $(X,\tau,\mathfrak{I})$ be a space with $\mathfrak{I} \sim \tau$. Then $\psi(A) = \cup\{\psi(U):U\in\tau,\ \psi(U)\text{-}A\in\mathfrak{I}\}$.

PROOF. Denote $\cup\{\psi(U):U\in\tau,\ \psi(U)\text{-}A\in\mathfrak{I}\}$ by $\psi'(A)$. Clearly, $\psi'(A) \subseteq \psi(A)$. Now let $x \in \psi(A)$ which implies there exists $U \in \tau(x)$ such that $U\text{-}A \in \mathfrak{I}$. By Theorem 3.1, $U \subseteq \psi(U)$, and $\psi(U)\text{-}A \subseteq [\psi(U)\text{-}U] \cup [U\text{-}A] \rightarrow \psi(U)\text{-}A \in \mathfrak{I}$. Hence $x \in \psi'(A)$ and the proof is complete.

Observe that if $f:X \rightarrow Y$ is an injection and $\mathfrak{J}$ is an ideal on Y, then $f^{-1}(\mathfrak{J}) = \{f^{-1}(J):J\in\mathfrak{J}\}$ is an ideal on X.

If $(X,\tau,\mathfrak{I})$ is a space and $\mathfrak{B}$ is a basis for a topology on X, then $\psi(\mathfrak{B})$ is a basis for a topology on X coarser than τ [7]. Denote by $<\psi(\mathfrak{B})>$ the topology generated by $\psi(\mathfrak{B}) = \{\psi(B):B\in\mathfrak{B}\}$.

Note that the previous theorem shows that $\psi(\mathfrak{I},\tau) = \psi(\mathfrak{I},<\psi(\mathfrak{I},\tau)(\tau)>)$, if $\mathfrak{I} \sim \tau$.

THEOREM 3.4. Let $(X,\tau,\mathfrak{I})$ and $(Y,\sigma,\mathfrak{J})$ be spaces with $f:(X,\tau) \rightarrow (Y,<\psi(\sigma)>)$ a continuous injection, $\mathfrak{J} \sim \sigma$, and $f^{-1}(\mathfrak{J}) \subseteq \mathfrak{I}$. Then $\psi(f(A)) \subseteq f(\psi(A))$ for every $A \subseteq X$.

PROOF. Let $y \in \psi(f(A))$ where $A \subseteq X$. Then by Theorem 3.3, there exists $V \in \sigma$ such that $y \in \psi(V)$ and $\psi(V)\text{-}f(A) \in \mathfrak{J}$. Now we have $f^{-1}(\psi(V)) \in \tau(f^{-1}(y))$ with $f^{-1}[\psi(V)\text{-}f(A)] \in \mathfrak{I} \rightarrow f^{-1}(\psi(V))\text{-}A \in \mathfrak{I} \rightarrow f^{-1}(y) \in \psi(A) \rightarrow y \in f(\psi(A))$, and the proof is complete.

THEOREM 3.5. Let $(X,\tau,\mathfrak{I})$ and $(Y,\sigma,\mathfrak{J})$ be spaces with $f:(X,<\psi(\tau)>) \rightarrow (Y,\sigma,\mathfrak{J})$ an open bijection, $\mathfrak{I} \sim \tau$, and $f(\mathfrak{I}) \subseteq \mathfrak{J}$. Then $f(\psi(A)) \subseteq \psi(f(A))$ for every $A \subseteq X$.

PROOF. Let $A \subseteq X$ and let $y \in f(\psi(A))$. Then $f^{-1}(y) \in \psi(A) \rightarrow$ there exists $V \in \tau$ such that $f^{-1}(y) \in \psi(V)$ and $\psi(V)\text{-}A \in \mathfrak{I}$ by Theorem 3.3. Now $f(\psi(V)) \in \sigma(y)$ and $f(\psi(V))\text{-}f(A) = f[\psi(V)\text{-}A] \in f(\mathfrak{I}) \subseteq \mathfrak{J}$. Thus $y \in \psi(f(A))$, and the proof is complete.

THEOREM 3.6. Let $f:(X,\tau,\mathfrak{I}) \rightarrow (Y,\sigma,\mathfrak{J})$ be a bijection with $f(\mathfrak{I}) = \mathfrak{J}$. Then the following are equivalent:

(1) f is a *-homeomorphism;

(2) $f(A^*) = [f(A)]^*$ for every $A \subseteq X$; and

(3) $f(\psi(A)) = \psi(f(A))$ for every $A \subseteq X$.

PROOF. (1) $\rightarrow$ (2). Let $A \subseteq X$. Assume $y \notin f(A^*)$. This implies $f^{-1}(y) \notin A^*$, and hence there exists $U \in \tau(f^{-1}(y))$ such that $U \cap A \in \mathfrak{I}$. Consequently $f(U) \in \sigma^*(y)$ and $f(U) \cap f(A) \in \mathfrak{J} \rightarrow y \notin f(A)^*(\mathfrak{J},\sigma^*) = f(A)^*(\mathfrak{J},\sigma)$. Thus $[f(A)]^* \subseteq f(A^*)$.

Now assume $y \notin [f(A)]^*$. This implies there exists a $V \in \sigma(y)$ such that $V \cap f(A) \in \mathfrak{J} \rightarrow f^{-1}(V) \in \tau^*(f^{-1}(y))$ and $f^{-1}(V) \cap A \in \mathfrak{I} \rightarrow f^{-1}(y) \notin A^*(\mathfrak{I},\tau^*) = A^*(\mathfrak{I},\tau) \rightarrow y \notin f(A^*)$. Hence $f(A^*) \subseteq [f(A)]^*$ and (2) holds.

(2) $\rightarrow$ (3). Let $A \subseteq X$. Then $f(\psi(A)) = f[X\text{-}(X\text{-}A)^*] = Y\text{-}f(X\text{-}A)^* = Y\text{-}(Y\text{-}f(A))^* = \psi(f(A))$.

(3) $\rightarrow$ (1). Let $U \in \tau^*$. Then $U \subseteq \psi(U)$ by Theorem 3.1 $\rightarrow f(U) \subseteq f(\psi(U)) = \psi(f(U)) \rightarrow f(U) \in \sigma^*$, and hence $f:(X,\tau^*) \rightarrow (Y,\sigma^*)$ is open. Similarly, $f^{-1}:(Y,\sigma^*) \rightarrow (X,\tau^*)$ is open and f is a *-homeomorphism.

DEFINITION. A function $f:(X,\tau,\mathfrak{I}) \rightarrow (Y,\sigma,\mathfrak{J})$ will be called a *ψ-homeomorphism with respect to τ, $\mathfrak{I}$, σ, and $\mathfrak{J}$* (simply a *ψ-homeomorphism* when no ambiguity is present) iff $f:(X,<\psi(\tau)>) \rightarrow (Y,<\psi(\sigma)>)$ is a homeomorphism.

THEOREM 3.7. Let $(X,\tau,\mathfrak{I})$ be a space, then $<\psi(\tau^*)> = <\psi(\tau)>$.

PROOF. Note that for every $U \in \tau$ and for every $I \in \mathfrak{I}$, we have $\psi(U\text{-}I) = \psi(U)$. Consequently, $\psi(\beta) = \psi(\tau)$ and $<\psi(\beta)> = <\psi(\tau)>$. It follows directly from Theorem 11 of [7] that $<\psi(\beta)> = <\psi(\tau^*)>$, hence the theorem is proved.

Our next theorem is the main theorem of this section.

THEOREM 3.8. (Lifting Theorem). Let $f:(X,\tau,\mathfrak{I}) \rightarrow (Y,\sigma,\mathfrak{J})$ be a bijection with $f(\mathfrak{I}) = \mathfrak{J}$.

(1) If f is a *-homeomorphism, then f is a ψ-homeomorphism.

(2) If $\mathfrak{I} \sim \tau$, $\mathfrak{J} \sim \sigma$, and f is a ψ-homeomorphism, then f is a *-homeomorphism.

PROOF. (1) Assume $f:(X,\tau^*) \to (Y,\sigma^*)$ is a homeomorphism, and let $\psi(U)$ be a basic open set in $<\psi(\tau)>$ with $U \in \tau$. Then $f(\psi(U)) = \psi(f(U))$ by Theorem 3.6 $\to$ $f(\psi(U)) \in \psi(\sigma^*)$, but $<\psi(\sigma^*)> = <\psi(\sigma)>$ by Theorem 3.7. Thus $f:(X,<\psi(\tau)>) \to (Y,<\psi(\sigma)>)$ is open. Similarily, $f^{-1}:(Y,<\psi(\sigma)>) \to (X,<\psi(\tau)>)$ is open and f is a ψ-homeomorphism.

(2) Assume f is a ψ-homeomorphism, then $f(\psi(A)) = \psi(f(A))$ for every $A \subseteq X$ by Theorems 3.4 and 3.5. Thus f is a *-homeomorphism by Theorem 3.6, and the proof is complete.

The hypotheses of $\mathfrak{I} \sim \tau$ and $\mathfrak{J} \sim \sigma$ are necessary in (2) of the above theorem as the following example shows.

EXAMPLE. Let X be the natural numbers. Denote by I_n the initial set of natural numbers 1 to n; i.e., $I_n = \{1,2,\ldots,n\}$. Let τ denote the topology $\{\emptyset,X\} \cup \{I_n : n \in X\}$. Let $\mathfrak{I}_f$ denote the ideal of finite subsets of X, and observe that $\psi(A) = X$ for every $A \subseteq X$ so that $<\psi(\tau)>$ is indiscrete. Also observe that τ^* is the discrete topology on X. Let Y denote the natural numbers and let σ be the indiscrete topology on Y, and consider the identity function i: $(X,\tau,\mathfrak{I}_f) \to (Y,\sigma,\mathfrak{I}_f)$. Clearly i is a ψ-homeomorphism since $<\psi(\tau)>$ and $<\psi(\sigma)>$ are both indiscrete. However, σ^* is the co-finite topology and hence i is not a *-homeomorphism.

Note that in the above example we have $\mathfrak{I}_f$ not compatible with τ but we do have $\mathfrak{I}_f \sim \sigma$ showing that compatibility cannot be relaxed in the hypotheses of Theorem 3.8, (2), on either the domain or range. Also note that the compatibility hypothesis $\mathfrak{J} \sim \sigma$ cannot be relaxed to the weaker conditon of $\beta = \sigma^*$ in the range.

The next example shows that the hypothesis $f(\mathfrak{I}) = \mathfrak{J}$ in Theorem 3.8, (1), cannot be relaxed to $f(\mathfrak{I}) \subseteq \mathfrak{J}$.

EXAMPLE. Let $X = \{0, 1\}$, $\tau = \{\emptyset, X, \{0\}\}$, $\sigma = \{\emptyset, X\}$, $\mathfrak{I} = \{\emptyset\}$, $\mathfrak{J} = \{\emptyset, \{1\}\}$ and $f:(X,\tau,\mathfrak{I}) \to (Y,\sigma,\mathfrak{J})$ be the identity function. Clearly, $f(\mathfrak{I}) \subseteq \mathfrak{J}$, $\mathfrak{I} \sim \tau$, and $\mathfrak{J} \sim \sigma$. It is also easily seen that $\tau^*(\mathfrak{I}) = \tau = \sigma^*(\mathfrak{J})$ and, hence, f is a *-homeomorphism. However, $<\psi(\mathfrak{I},\tau)(\tau)> = \tau \neq \sigma = <\psi(\mathfrak{J},\sigma)(\sigma)>$ so that f is not a ψ-homeomorphism.

As an application of the Lifting Theorem, we will prove a theorem partially due to Crossley and Hildebrand ([3] and [4]), as mentioned earlier. First we need a preliminary result.

THEOREM 3.9. Let f: $(X,\tau) \to (Y,\sigma)$ be a semihomeomorphism, then f is a δ-homeomorphism.

PROOF. Let $f:(X,\tau) \to (Y,\sigma)$ be a semihomeomorphism. It suffices to show that f preserves regular open sets. If $V \subseteq X$ is regular open, V is semiclopen (i.e. both semiopen and semiclosed in the sense that X-V is also semiopen) so that f(V) is semiclopen. Then Int(f(V)) is regular open and Cl(f(V)) = Cl(Int(f(V))) and so Cl(f(V))-Int(f(V)) is nowhere dense. Since semihomeomorphisms preserve nowhere dense sets [4], B-A is nowhere dense with $B = f^{-1}(Cl(f(V)))$ and $A = f^{-1}(Int(f(V)))$. Thus, $Int(B)-Cl(A) = \emptyset$ showing that $Int(B) \subseteq Int(Cl(A))$. But Int(f(V)) is semiclopen $\to$ A is semiclopen and Int(A) = Int(Cl(A)). Therefore, $Int(B) \subseteq Int(Cl(A)) = Int(A) \subseteq V \subseteq Int(B)$ and $V = Int(A) \subseteq A$. This yields $f(V) \subseteq f(A) = Int(f(V))$ so that f(V) is open. Since f(V) is also semiclosed, it must be regular open.

THEOREM 3.10. Let $f:(X,\tau) \to (Y,\sigma)$ be a bijection. Then f is a semihomeomorphism iff f is an α-homeomorphism.

PROOF. NECESSITY. Let $\mathcal{N}(\tau)$ and $\mathcal{N}(\sigma)$ denote the ideals of nowhere dense sets with respect to τ and σ, respectively. It is well known [21] that $\mathcal{N}(\tau) \sim \tau$ and $\mathcal{N}(\sigma) \sim \sigma$. Also observe that $<\psi(\tau)> = \tau_s$ and $<\psi(\sigma)> = \sigma_s$ [7]. Thus if f is a semi-homeomorphism, it follows from Theorem 3.9 that f is a ψ-homeomorphism, and it follows from the Lifting Theorem that f is an α-homeomorphism.

SUFFICIENCY. Let $f:(X,\tau) \to (Y,\sigma)$ be an α-homeomorphism. Then $f^{\alpha}:(X,\tau^{\alpha}) \to (Y,\tau^{\alpha})$ is a homeomorphism where $f^{\alpha}(x) = f(x)$ for each $x \in X$. Since f^{α} and $(f^{\alpha})^{-1}$ preserve semiopen sets and the semiopen subsets of (X,τ^{α}) and (Y,σ^{α}) are precisely those of (X,τ) and (Y,σ) respectively, f and f^{-1} also preserve semiopen sets.

4. α-TOPOLOGICAL PROPERTIES

By Theorem 3.10 a property P is a semitopological property if and only if P is an α-topological property and it is clear that the latter holds if and only if (X,τ) and (X,τ^{α}) both have P whenever either does. Andrijevic' [1] has shown that for each $A \subseteq X$, $Int(Cl(A))) = Int^{\alpha}(Cl^{\alpha}(A))$. It follows that (X,τ) and (X,τ^{α})

share the same nowhere dense and meager sets and that Baireness is an α-topological property. Since (X,τ) and (X,τ^{α}) also share the same dense sets, resolvability and separability are also α-topological properties. In the literature, many authors have isolated semitopological properties which in fact were semiregular properties and hence by Corollary 2.5 are α-topological and hence semitopological. The examples below show that Baireness, resolvability, and separability are semitopological properties which are not semiregular.

EXAMPLE. The space $X = \{1,2,3,...\}$ with the cofinite topology is not Baire since finite sets are nowhere dense and X is meager. Yet the semiregularization of X is Baire since it is indiscrete.

EXAMPLE. Two-point Sierpinski space is not resolvable whereas its semiregularization is indiscrete and thus resolvable.

EXAMPLE. The uncountable set of real numbers R with the cocountable topology is not separable, but its indiscrete semiregularization is separable.

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