Internat. J. Math. & Math. Sci.
VOL. 13 NO. 3 (1990) 611-616

REMARKS ON QUADRATIC EQUATIONS IN BANACH SPACE

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(Received May 12, 1988 and in revised form May 29, 1989)

ABSTRACT. We find new sufficient conditions for the uniqueness of solutions of quadratic and in general multilinear equations in a Banach space X, by assuming the existence of a certain limit of linear operators in a suitable subspace of X.

1. INTRODUCTION.

Consider the quadratic equation

$$x = y + B(x, x) \tag{1.1}$$

in a Banach space X, where $y \in X$ is fixed and B is a bilinear operator on X [1], [2]. We can assume without loss of generality that B is symmetric, otherwise we can replace B in (1.1) by

$$\overline{B}(x, y) = \frac{1}{2}(B(x, y) + B(y, x)) \text{ for all } x, y \in X$$

which is a symmetric bilinear operator on X and bounded if B is.

Using some ideas given in [3], [4], and [5] for the linear case, we derive some new existence and uniqueness results for the solutions x of (1.1) similar to the ones in [4] for the linear case and different than the results in [6].

Moreover, under the assumption that for all $z_1, z_2 \ldots, z_n, x \in X$

$$\lim_{n\to\infty} B(z_1)B(z_2) \ldots B(z_n)(x) \tag{1.2}$$

exists, where $B(z_i)$ denotes a linear operator on X for $i = 1,2, \ldots ,n$ such that

$$B(z_i)(x) = B(z_i, x) = B(x, z_i),$$

we give more insight into the behavior of equation (1.1).

Finally, it will be obvious from the proofs of the theorems that the results obtained here can be easily generalized to include multilinear equations [2], [7], [8].

Let $z_1, z_2, \ldots, z_n$ be fixed in X. If the limit in (1.2) exists as $n \to \infty$, then (1.2) defines a linear operator L, depending on the choice of $z_1, \ldots, z_n$, on X, given by

$$L(x) = \lim_{n\to\infty} B(z_1)B(z_2)\ldots B(z_n)(x). \tag{1.3}$$

We denote the domain of L by D(L) and the null space of L by N(L). By $x \in D(L)$ or $x \in N(L)$ we mean

$$L(x) \text{ exits or } L(x) = 0$$

where L is given by (1.3). Note that $N(L) \neq \emptyset$, since $0 \in N(L)$.

2. MAIN RESULTS.

The following is essentially Rall's theorem [6] proved in a different way which allows us to associate it with (1.3).

THEOREM 2.1. If the linear operator B(z) has no nonzero fixed point for all $z \in X$ then (1.1) has at most one solution.

PROOF. If (1.1) has no solution there is nothing to prove. Let $\bar{x}$ be a solution of (1.1), then any other solution x can be given by

$$x = \bar{x} + h, \text{ for some } h \in X.$$

We now have by (1.1),

$$\bar{x} + h = y + B(\bar{x} + h, \bar{x} + h)$$

or

$$B(z)(h) = h, \; z = 2\bar{x} + h.$$

By hypothesis h = 0. Therefore, the solution $\bar{x}$ is unique.

If the equation (1.1) has a solution $\bar{x}$ then the totality of the solutions is given by $x = \bar{x} + h$ where h is a fixed point of the linear operator B(z) with $z = 2\bar{x} + h$. The solution $\bar{x}$ is unique if h = 0. Every fixed point of the operator B(z) is a fixed point of the operator

$$L(x) = \lim_{n\to\infty} (B(z))^n(x), \text{ with } z = 2\bar{x} + h .$$

The equation

$$L(h) = h \text{ shows that } h = 0 \text{ if } X = N(L).$$

Thus, the solution $\bar{x}$ is unique in this case. Therefore, for this particular choice of L the condition $X = N(L)$ is a sufficient condition for the uniqueness of $\bar{x}$.

The assumptions that X is indeed a Banah space was not used in the derivation of the above results. It will be used, however, in the following results.

We now prove Thoerems 2.2-2.5 which are generalizations of the thoerems in [3], [5] and of the remarks 1-5 in [4].

THEOREM 2.2. Let w be a solution of (1.1). Define the sequence $\{x_n\}$, n = 1,2, ... by

$$x_n = y + B(x_{n-1}, x_{n-1}) \text{ for some } x_0 \in X. \tag{2.1}$$

Then $\{x_n\}$, n = 1, 2, ... converges to some point $v \in X$ if and only if

$$x_0 - w \in D(L),$$

where L is given by (1.3) for $z_n = z_{n-1} + w$, n = 1, 2, 3, ...

PROOF. We have

$$\begin{aligned} x_n - v &= (y + B(x_{n-1}, x_{n-1})) - (y + B(w, w)) \\ &= B(x_{n-1}, x_{n-1}) - B(w, w) \\ &= B(x_{n-1} + w, x_{n-1} - w) \\ &= B(x_{n-1} + w)B(x_{n-2} + w, x_{n-2} - w) \\ &\cdots\cdots \\ &= B(z_n)B(z_{n-1}) \ldots B(z_1)(x_0 - w), \end{aligned}$$

where $z_k = x_{k-1} + w$, k = 1, 2, ..., n. Now if $x_0 - w \in D(L)$ the limit on the right hand side of the above exists, therefore $\{x_n\}$, n=1,2, ... converges to some point $v \in X$.

Conversely, if $\{x_n\}$, n = 1, 2, ... converges to some $v \in X$ as $n \to \infty$ then

$$x_0 - w \in D(L) .$$

Sometimes this weaker condition sufficies, as indicatted next:

THEOREM 2.3. Let x_0 be an approximate solution of equation (1.1) in the sense that $x_0 - B(x_0, x_0) - y \in N(L)$, where L is given by (1.3) with $z_k = x_{k-1} + x_k$, k = 1,2,..., and the x_n's are given by (2.1).

Then every limit point of the sequence $\{x_n\}$, n = 1, 2, ... satisfies equation (1.1).

PROOF. As in Theorem 2.2,

$$x_{n+1} - x_n = B(z_n)B(z_{n-1})\ldots B(z_1)(x_0 - x_1)$$

where $z_k = x_{k-1} + x_k$, $k = 1, 2, \dots, n$.

Now if $x_n \to x$ for $n = n_i \to \infty$, then $x_{n+1} \to x$ for $n = n_i \to \infty$, and the result follows if we let $n_i \to \infty$ in $x_{n+1} = y + B(x_n, x_n)$.

Furthermore we can show:

THEOREM 2.4. If $x \in D(L)$ is a solution of (1.1), then every other solution w is in $D(L)$ where L is given by (1.3) with $z_n = x + w$, $n = 0, 1, 2, \dots$.

PROOF. Since $x = y + B(x, x)$ and $w = y + B(w, w)$ then $w - x$ is a fixed point of $B(x + w)$ and hence $w - x \in D(L)$. Therefore $w = (w - x) + x \in D(L)$.

The following result complements Theorem 2.2.

THEOREM 2.5. Equation (1.1) has a solution $x \in D(L)$ where L is as in Theorem 2.2 if and only if the sequence $\{t_n\} = \{s_0 + s_1 + \dots + s_n\}$ converges, where

$$s_0 = y,$$

$$s_k = \sum_{j=0}^{k-1} B(s_j, s_{k-j-1}), \quad k = 1, 2, \dots .$$

PROOF. The given sequence $\{t_n\}$, $n = 1, 2, \dots$ corresponds to $\{x_{n+1}\}$ with $x_0 = 0$. If it converges to some $x \in X$ then x is a solution of (1.1). By Theorem 2.2., $x \in D(L)$ (for $x_0 = 0$, $w = x$). Now since $x_0 = 0 \in D(L)$, the converse follows from Theorem 2.2.

3. APPLICATIONS.

We now complete this paper with an example. Let $X = \mathbb{R}^2$ and consider the bilinear operator on X given by

$$B(x,y) = \left\{ (x_1, x_2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \hline 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 \\ x_1 & -x_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 y_1 + x_2 y_2 \\ x_1 y_1 - x_2 y_2 \end{bmatrix}.$$

Note that B is symmetric bilinear operator on X. We consider the equation

$$x = 0 + B(x, x)$$

or

$$\begin{aligned} x_1 &= x_1^2 + x_2^2 \\ x_2 &= x_1^2 - x_2^2. \end{aligned} \qquad (3.1)$$

Define a norm on $\mathbb{R}^2$ by

$$||x|| = \max_{1 \leq i \leq 2} ||x_i|| .$$

Let $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$, then

$$B(z)B(x)(y) = \begin{bmatrix} z_1 & z_2 \\ z_1 & -z_2 \end{bmatrix} \begin{bmatrix} x_1y_1 + x_2y_2 \\ x_1y_1 - x_2y_2 \end{bmatrix}$$

$$= \begin{bmatrix} z_1x_1y_1 + z_1x_2y_2 + z_2x_1y_1 - z_2x_2y_2 \\ z_1x_1y_1 + z_1x_2y_2 - z_2x_1y_1 + z_2x_2y_2 \end{bmatrix} .$$

If we apply the above for $z_i = \begin{bmatrix} z_i^i \\ z_i^i \end{bmatrix}$ $i = 1, 2, \ldots, n$, then $B(z_1)B(z_2) \ldots B(z_n)(x)$ will be of the form

$$v_n = \begin{bmatrix} \sum_{k=1}^{2^n} c_1^k c_2^k \ldots c_{2^{n+1}}^k \\ \sum_{k=1}^{2^n} d_1^k d_2^k \ldots d_{2^{n+1}}^k \end{bmatrix}$$

where d_m^k, $c_m^k \in \{x, z_i\}$, $i = 1, 2, \ldots, n$, $k = 1, 2, \ldots, 2^n$, $m = 1, 2, \ldots, 2^{n+1}$.

Now let us restrict X to $[0, \frac{1}{4}] \times [0, \frac{1}{4}]$, then

$$||v_n|| \leq 2^n \cdot (\tfrac{1}{4})^{n+1} = \frac{1}{2^{n+1}} \to 0 \text{ as } n \to \infty;$$

therefore

$$L(x) = 0 \text{ for all } x \in [0, \tfrac{1}{4}] \times [0, \tfrac{1}{4}] \equiv \overline{X}$$

so $\overline{X} = N(L)$ and equation (3.1) has a unique solution in $\overline{X}$, namely $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

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