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# A PROOF OF POLLACZEK-SPITZER IDENTITY

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**ABSTRACT.** In this note we derive a proof of Pollaczek-Spitzer identity using a generalization of Takacs ballot theorem.

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## 1. INTRODUCTION.

Consider the following generalization of Takacs ballot theorem (Takacs [1]): Suppose $k_1, k_2, \ldots, k_n$, are non-negative integers with sum $k < mn$ for some integer $m$ and let $n_r$ be the number of cyclic permutations $(k_{i_1}, k_{i_2}, \ldots, k_{i_n})$ of $(k_1, k_2, \ldots, k_n)$ such that $k_{i_1} + k_{i_2} + \ldots + k_{i_j} \le jm - r$ for all $j = 1, 2, \ldots, n$, with equality holding for at least one of these $j$'s, $r = 1, 2, \ldots, m$. Then

$$\sum_{r=1}^{m} rn_r = nm - k \tag{1.1}$$

On setting $r_i = m - k_i$, we get the following generalization: Let $r_1, r_2, \ldots, r_n$ be integers with sum $s \ge 1$ and let $n_r$ be the number of cyclic permutations in which all the partial sums are greater or equal to $r$ with at least one sum equal to $r$. Then

$$\sum_{r=1}^{s} rn_r = s \tag{1.2}$$

**PROOF of (1.1).** Consider $n$ boxes arranged in a circle and numbered 1 to $n$ in the clockwise direction. Initially box $i$ contains $k_i$ balls. Starting from box $n$ search the boxes in the anti-clockwise direction and should a box contain $m + r$ balls for some $r > 0$, then remove $r$ balls from the box containing these $m + r$ balls and place them in the box that follows immediately in the anti-clockwise direction. Repeat the above steps until the number of balls contained in each box is less than or equal to $m$. Let $B_i$ be the number of balls contained in box $i$ after the re-allocations as specified are completed and let $n_i$ be the number of integers among $B_1, B_2, \ldots, B_n$ which are equal to $m - i$, $i = 0, 1, \ldots, m$. Since $\Sigma (m - i)n_i = k$ and $\Sigma n_i = n$, we have $\Sigma in_i = nm - k$.

Let $k_{n+i} = k_i$ and $S_{ij} = k_i + k_{i+1} + \ldots + k_{i+j}$, $i, j = 1, 2, \ldots, n$. Then $B_i = m - r$, $1 \le r \le m$, if and only if $S_{ij} \le jm - r$ for all $j$ with at least one index $t$ for which $S_{it} = tm - r$. To prove this assume without loss of generality that $i = 1$. Suppose $B_1 = m - r$, $1 \le r \le m$, and $S_{1t} > tm - r$ for some $t \ge 2$. Then we must have $B_t = B_{t-1} \dots = B_2 = m$ and $B_1 > m - r$, a contradiction. So $S_{ij} \le jm - r$ for all $j \ge 1$. Suppose $S_{ij} < jm - r$ for all $j$. Then we must have $k_1 \le m$ and $k_i \le m$ for all $i \ge 2$, which implies that $B_1 < m - r$, a contradiction. Now (1.1) follows immediately.

## 2. POLLACZEK-SPITZER IDENTITY.

Using (1.2), we give a proof of the well-known Pollaczek-Spitzer identity (2.1). This proof appears to be new. To keep the arguments simple, we consider integer-valued random variables only.

**THEOREM.** Let $X_i$, $i = 1, 2, \dots$, be an infinite sequence of independent and identically distributed integer-valued random variables; $S_n = X_1 + X_2 + \dots + X_n$; $m_{i,j}$ and $M_{i,j}$, the minimum and maximum respectively of $X_i, X_i + X_{i+1}, \dots, X_i + X_{i+1} + \dots + X_{i+j}$;

$$F_i = \sum_{s=1}^{\infty} \exp(-\lambda s) P\{S_i = s\}/i, \qquad i = 1, 2, \dots,; \qquad \lambda \ge 0$$

$$G_i = \sum_{s=1}^{\infty} \exp(-\lambda s) P\{M_{1,i-1} \le 0, S_i = s\}, \qquad i = 1, 2, \dots; \qquad \lambda \ge 0$$

$$F = \sum_{i=1}^{\infty} t^i F_i, \quad G = \sum_{i=1}^{\infty} t^i G_i, \quad 0 < t < 1$$

Then

$$F = -\log(1 - G) \tag{2.1}$$

**PROOF.** By (1.2), we have

$$\sum_j j\, P\{m_{1,n} = j \mid S_n = s\} = s/n \tag{2.2}$$

provided the conditional probability exists. Now for $r < s$,

$$\{m_{1,n} = r \mid S_n = s\} = U[\{(m_{1,i} = r \mid S_i = s - t) \cap (S_i = s - t)\} \cap \{m_{i+1,n} = t \mid S_n - S_i = t\}] \tag{2.3}$$

where the union is over all $i \ge 1$ and all $1 \le t \le s - r$. Also note the easily verifiable duality property

$$P\{m_{1,n} = s | S_n = s\} = P\{M_{1,n-1} \le 0 | S_n = s\} \tag{2.4}$$

Consequently, using (2.3) and (2.4), we have for $r < s$,

$$P\{m_{1,n} - s \mid S_n = s\} = P\{m_{1,i} = r \mid S_i = s - t\} P\{S_i = s - t\} P\{M_{i+1,n-1} \le 0 \mid S_n - S_i = t\} \tag{2.5}$$

So multiplying (2.5) by $r \le s - 1$, adding the quantity $P\{m_{1,n} = s \mid S_n = s\}$ to both sides of the equation, and summing, we get, by (2.2),

$$s/n = s\, P\{M_{1,n-1} \le 0 \mid S_n = s\} + \sum_{i,t} \frac{(s-t)}{i} P\{S_i = s - t\} P\{M_{i+1,n-1} \le 0 \mid S_n - s_i = t\}$$

which implies that

$$s\, P\{S_n = s\}/n = s\, P\{M_{1,n-1} \le 0 \mid S_n = s\} P\{S_n = s\}$$
$$+ \sum_{i,t} \frac{(s-t)}{i} P\{S_i = s - t\} P\{M_{i+1,n-1} \le 0 \mid S_n - S_i = t\} P\{S_n - S_i = t\} \tag{2.6}$$

Then multiplying (2.6) by $\exp(-\lambda s)$ and summing over all $s \ge 1$, we obtain

$$F_n' = G_n' + \sum_{i=1}^{n-1} F_i' G_{n-i} \tag{2.7}$$

where

$F_i' = dF_i/d$ and $G' = dG_i/d$. Multiplying (2.7) by $t^n$, $0 < t < 1$, and summing over $n = 1, 2, \dots$, we have

$$\sum_i t^i F_i = \left(\sum_i t^i G_i'\right)/(1-G)$$

So integrating, we get the identity (2.1). (Let $\lambda \rightarrow \infty$ to show that the arbitrary constant is zero.)

Proofs of (2.1) and other closely related results can be found in the references [2] - [10].

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