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GENERALIZED SUM-FREE SUBSETS

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ABSTRACT. Let $F = \{A(i): 1 \leq i \leq t, t \geq 2\}$, be a finite collection of finite, pairwise disjoint subsets of Z^+. Let $S \subset R\backslash\{0\}$ and $A \subset Z^+$ be finite sets. Denote by $S^A = \{\sum_{i=1}^{a} s_i : a \in A, S_i \in S$, the s_i are not ncesarily distinct $\}$. For S and F as above we say that S is F-free if for every $A(i), A(j) \in F$, $i \neq j$, $S^{A(i)} \cap S^{A(j)} = \phi$.

We prove that for S and F as above, S contains an F-free subset Q such that $|Q| \geq c(F)|S|$, when c(F) is a positive constant depending only on F.

This result generalizes earlier results of Erdos [3] and Alon and Kleitman [2], on sum-free subsets. Several possible extensions are also discussed.

1. INTRODUCTION.

A set S of integers is called <u>sum-free</u> if $(S+S) \cap S = \emptyset$, i.e. if there are not (not necessarily distinct), $a,b,c \in S$ such that: $a + b = c$. There is a considerable amount of results concerning sum-free subsets, not only within the integers, but also in the context of abelian groups. The monumental survey by Wallis and Street [1] is recommended for that purpose. Recently Alon and Kleitman [2] proved (among many other interesting results) the following theorem.

THEOREM A:[2]. Any finite set B of nonzero reals contains a sum-free subset A of cardinality $|A| > |B|/3$.

This is a slight improvement of an old result of Erdos [3]. Here we consider the more general problem mentioned in the abstract. Let us first recall the exact formulation of the generalized sum-free subset problem. Let $F = \{A(i): 1 \leq i \leq t, t \geq 2\}$, be a finite collectiion of finite, pairwise disjoint, subsets of Z^+. Let $A \subset Z^+$ and $S \subset R \setminus \{0\}$ be finite sets. Denote by $S^A = \{\sum_{i=1}^{a} s_i$: $a \in A$, $s_i \in S$, the s_i are not necessarily distinct }. For S and F as defined above we say that S is F-free if for every $A(i), A(j) \in F$, $i \neq j$, $S^{A(i)} \cap S^{A(i)} = \emptyset$. Clearly for $F = \{\{1\}, \{2\}\}$ this is just the case of sum-free subsets investigated in [2] and [3].

Erdos as well as Alon and Kleitman gave a probabilistic proof of Theorem A. We shall give a proof of Theorem A in the case of integers, avoiding the probabilistic tools and using a double-counting instead. Our main goal however is to generalize both the content and the proof technique of Theorem A to an arbitrary collection F as explained above. We shall combine the probabilistic ideas of [2] together with an observation on non-intersecting subintervals of the unit interval $[0, 1)$. We prove the following:

THEOREM 1.1. Let S be a finite subset of $R\setminus\{0\}$. Let $F = \{A(i): 1 \le i \le t, \ t \ge 2\}$ be a finite scollection of finite, pairwise disjoint, subsets of Z^+, then S contains an F-free subset Q of cardinality $|Q| \ge c(F)|S|$, when $c(F)$ is a positive constant depending on F only.

2. PROOFS OF THE THEOREMS.

We first give a non-probabilistic proof of Theorem A in the case of integers.

PROOF OF THEOREM A. Let $B = \{b_1, b_2, \ldots, b_n\}$ be a set of integers. Let $p = 2k + 2$ be a prime number such that: $p > 2 \max_{1 \le i \le n} |b_i|$, and put $C = \{k+1, k+2, \ldots, 2k+1\}$. Observe that C is a sum-free subset of the cyclic group Z_p, and that $|C|/(p-1) = (k+1)/(3k+1) > 1/3$.

For any x, $1 \le x \le p-1$, define $d_i(x) = d_i$ by $d_i \equiv xb_i \pmod p$, $0 \le d_i < p$. Clearly, for every fixed i, $1 \le i \le n$, as x ranges over all numbers $1,2..,p-1$, d_i ranges over all nonzero elements of Z_p.

Now we use double-counting instead of the probablistic argument. For every b_i, $1 \le i \le n$, let $t(b_i) = |\{x: d_i = xb_i \in C\}|$. For every x, $1 \le x < p$, let $r(x) = |\{i: d_i = xb_i \in C\}|$. By double-counting we have $\sum_{i=1}^{n} t(b_i) = \sum_x r(x)$, but by the choice of P, $t(b_i) = k + 1$ for $1 \le i \le n$. Hence for some $1 \le x \le p-1$, $r(x) \ge n(k+1)/(p-1) = n(k+1)/(3k+1) > n/3$. Consequently there is a subset A of B, of cardinality $|A| > |B|/3$ such that $xa \pmod p \in C$, for all $a \in A$. This subset A is sum-free, since if $a + b = c$, for some, $a,b,c \in A$ then $xa + xb = xc \pmod p$, which is impossible because C is sum-free in Z_p. Q.E.D.

The proof of Theorem A rests on the basic idea which is to find a "large" sum-free subinterval of Z_p and to map the set, under consideration, onto Z_p such that a large portion of it, is mapped onto the sum-free subinterval. We shall apply such an idea in the following proof of Theorem 1.

PROOF OF THEOREM 1. Let $A(i) \in F$ be a finite set of positive integers and order them in an ascending order, such that $A(i) = \{a(i,1) < a(i,2) < \ldots < a(i,n_i)\}$. Denote by $\delta = \delta(F) = \min\{a(i,1): 1 \le i \le n\}$. Consider the sequence $B(F)$ of the largest elements of each set A_i, namely $B(F) = \{a(i,n_i): 1 \le i \le n\}$. Denote by $L_1(F)$ resp. $L_2(F)$ the largest resp. the second largest element of $B(F)$. Finally let $r = r(F) = \max\{x \in R: L_2(F) < x \le L_1(F),\ x/L_2(F) \le \min\{a(k,l)/a(i,j): i \ne k,\ a(k,l) > a(i,j)\}\}$.

We are going to show that $c(F)$, the constant in Theorem 1, is at least $c(F) \ge (r - L_2(F))/(rL_1(F) - \delta L_2(F)) > 0$. First we show that the interval $[\alpha,\beta) \subset [0,1]$ is F-free with respect to addition modulo-1, with the choice $:\alpha = L_2(F)/(rL_1(F) - \delta L_2(F))$, and $\beta = r/(rL_1(F) - \delta L_2(F))$. From the definitions $rL_1(F) - \delta L_2(F) > 0$, hence $0 < \alpha < \beta$. Moreover, since $\delta L_2(F) \le r(L_1(F) - 1)$, we infer that $\beta < 1$, hence it follows that $[\alpha,\beta) \subset [0,1]$. Now suppose there are

$a(i,j) \in A(i)$ and $a(k,l) \in A(k)$, such that $a(k,l) > a(i,j)$ and $a(k,l)[\alpha,\beta)(\mathrm{mod}\ 1) \cap a(i,j)[\alpha,\beta)(\mathrm{mod}\ 1) = 0$. ($c[\alpha,\beta) =: [c\alpha,c\beta)$). Observe that because of the specific choice of the parameters α, β and since $\beta L_1(F) = 1 + \delta\alpha$ as can be seen by a direct verification, we actually must have an intersection even without (mod 1) consideration, namely we must have $a(k,l)\ [\alpha,\beta) \cap a(i,j)[\alpha,\beta) = 0$. Hence it follows that $a(i,j)\beta > a(k,l)\alpha$, which in turn implies that $r/L_2(F) = \beta/\alpha > a(k,l)/a(i,j) > 1$, a contradiction to the choice of $r=r(F)$. Hence with respect to addition modulo-1, $[\alpha,\beta)$ is indeed F-free. Now let $\mu = \min_{1 \le i \le n} |s_i|$. We choose, randomly, a real number x according to a uniform distribution on the interval I, defined by $I = [1/\mu,\ n\theta(F)/\mu]$, when $\theta(F)$ is a large constant depends only on F, and compute the numbers $d_i(x) =: d_i = (xs_i)$ (mod 1). Observe that for fixed i, $1 \le i \le n$, every subinterval of I of length $1|s_i|$ is mapped onto $[0,1)$, hence for large $\theta(F)$ the expected number of d_i-s that belongs to $[\alpha,\beta)$ is more than $(\beta-\alpha)\ (n-1)$ and hence there is an x and a subset Q of at least $(\beta-\alpha)n$ members of S such that (xq) (mod 1) $\in [\alpha,\beta)$ for each $q \in Q$. But now we are done, because $[\alpha,\beta)$ is F-free with respect to addition modulo 1 and also $\beta-\alpha = (r-L_2(F))/(rL_1(F) - \delta L_2(F))$ as needed.

As a demonstration of the parameter involved let $A(1) = \{1,4,7\}$, $A(2)=\{2,5,8\}$, $A(3) = \{3,6,9\}$ and $F = \{A(1), A(2), A(3))$. Then $\delta(F) = 1$, $B(F) = 7,8,9\}$, $L_1(F) = 9$, $L_2(F) = 8$, $r(F) = 9$ and $c(F) = (9-8)/(9*9 - 1*8) = 1/73$. Moreover the interval $[8/73, 9/73)$ is F-free with respect to addition modulo-1.

3. EXTENSIONS AND VARIATIONS.

1. A set of linear equations over Z.

A natural question that can be asked now is, what can we say, for example, about the following set of linear equations:

$$5a + 3b + 2c = 9x$$
$$3a + 3b + 3c + d = 8x + y$$
$$4a + 2b + 2c + 2d = 6x + 2y + z$$

One can easily check that these equations are special cases of the F-free problem. Indeed we only have to define $A(1)=\{10\}$, $A(2)=\{9\}$, and $F = \{A(1),A(2)\}$. Any F-free subset contains no solution of any of the above equations. This observation can be set in a general form as follows.

THEOREM 3.1. Let $\sum_{i=1}^{u_j} a(i,j)x_i = \sum_{i=1}^{m_j} b(i,j)y_i$, $1 \le j \le n$, be a set of n linear equations with positive integer coefficients. Suppose further that for any u and v, $\sum_{i=1}^{n_u} a(i,u) \ne \sum_{i=1}^{m_v} b(i,v)$, then any set S of nonzero reals contains a subset Q, of cardinality $|Q| > c|S|$, when c is a positive constant depending only on the sums of the coefficients of the equations, and such that no linear equation of the prescribed set is solvable within Q.

PROOF. Observe that we only have to define $A(1)=\{\sum_{i=1}^{n_j} a(i,j),\ 1 \le j \le n\}$, $A(2) = \{\sum_{i=1}^{m_j} b(i,j),\ 1 \le j \le n\}$, and $F = \{A(1), A(2)\}$. By Theorem 1 we are done, because any F-free subset of S contains no solution of any of the equations.

2. A set of linear equations over Q.

Lets' have a look at the following linear equations over Q:

$$5a/3 + 2b/9 = 2x/7 + 5y/14$$
$$a/3 + b/3 + c/3 = x/2 + y/2.$$

There is an essential difference between them. The first one can be transformed into the equation

$$210a + 28b = 36x + 45y.$$

Clearly $210 + 28 = 238 \neq 81 = 36 + 45$. Hence Theorem 3.1 can be applied here with $A_1 = \{238\}$, and $A_2 = \{81\}$. The second equation can be transformed into $2a + 2b + 2c = 3x + 3y$, but now $2 + 2 + 2 = 6 = 3 + 3$, and we can't use Theorem 1 or Theorem 3.1, because the condition $A(1) \cap A(2) = \emptyset$ is violated. However from the examples given above it is clear that Theorem 3.1 remains valid in the more general situation of positive rational coefficients, and that a generalization of Theorem 1 to the case of rationals is possible. We omit the quite obvious details.

3. There is still the question of what can be said, if anything, in the case when for some $A(i), A(j) \in F$, $A(i) \cap A(j) \neq \emptyset$, e.g $a + b = 2c$ or $a + 2b + 3c + 4d = 10e$ etc. We hope to comment about such problems in Wallis, Street, and Wallis [4] under the frame of independent sets of hypergraphs. One can see that even the simplest case $a + b = 2c$, is closely related to the well known Theorem of Szemeredi on arithmetic progressions. So it is unreasonable to expect that results like Theorem 1 and Theorem 3.1 can hold in this case.

4. TWO OPEN PROBLEMS.

1. The most interesting open problem is that of determining the best possible constant in Theorem 1, with respect to a given set F, and in particular for the case of sum-free sets to prove or disprove that 1/3 is the best possible constant.

2. Another interesting problem is to obtain a constant better than 1/3 in the case when S is a set of squares of nonzero integers. Here we hve to avoid $a^2 + b^2 = c^2$. Clearly we can't hope for a constant better than 2/3 because we can take a large set consisting of arbitrarily large pythagorian's triples.

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