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ON THE JOINT NUMERICAL STATUS AND TENSOR PRODUCTS

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ABSTRACT. We prove a result on the joint numerical status of the bounded Hilbert space operators on the tensor products. The result seems to have nice applications in the multiparameter spectral theory.

KEY WORDS AND PHRASES. Joint numerical status, Hilbert tensor product, Multiparameter spectral theory, Joint Numerical range, Maximal numerical range.
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1. INTRODUCTION AND DEFINITIONS.

In a recent paper Buoni and Wadhwa [1] introduced the concept of the joint numerical status, and have shown that the joint numerical status of commuting normal operators equals the closure of their joint numerical range. Based upon the elegant work of Stampfli [2] on the norm of the inner derivation acting on the Banach algebra of all bounded linear operators on a Hilbert space, Fong [3] introduced the concept of essential maximum numerical range to derive the norm of an inner derivation on the Calkin algebra, and proved several interesting results. The results of Fong seem to hold for the case of the joint maximum numerical status as well.

On the other hand, Dash [4] has proved that the joint numerical range of the tensor product of operators is equal to the Cartesian product of their numerical ranges. Motivated by the work of Dash, we shall prove an analogous result for the joint numerical status of the tensor product of operators. It seems that results obtained in this paper have nice applications in the multiparameter spectral theory [5].

Let H be a complex Hilbert space, and let L(H) denote the algebra of all bounded linear operators on H. The algebra L(H) is a Banach space with respect to an operator norm (See Debnath and Mikusinski [6]). In the sequel, we need the following definitions:

DEFINITION 1.1. For $A = (A_1, \ldots A_n) \in L(H)^n$, the joint numerical range of A is denoted by W(A) and defined by

$$W(A) = \{(\langle A_1x, x\rangle, \ldots, \langle A_nx, x\rangle): ||x|| = 1, x \in H\}. \tag{1.1}$$

For the details about the joint numerical ranges, see references [1], [7] and [8].

DEFINITION 1.2. Let $C^*(L(H))$ denote the C^*-algebra generated by I and $A_1,\dots,A_n$. Then, for $A = (A_1,\dots,A_n) \varepsilon C^*(L(H))$, the joint numerical status of A is denoted by S(A) and defined by

$$S(A) = \{(f(A_1),\dots,f(A_n)): f(I) = ||f|| = 1\}, \tag{1.2}$$

where f is a state on $C^*(L(H)$.

We note that

$$S(A_1,\dots,A_n) \subseteq S(A_1) \times\dots\times S(A_n). \tag{1.3}$$

DEFINITION 1.3. Let $H_1,\dots,H_n$ be complex Hilbert spaces with inner products $\langle , \rangle_1,\dots,\langle , \rangle_n$, respectively. Let $\hat{H} = H_1 \hat{\otimes} \dots \hat{\otimes} H_n$ denote the Hilbert space completion of $H = H_1 \otimes \dots \otimes H_n$ with respect to inner product $\langle , \rangle$ which is defined by

$$\langle x,y\rangle = \langle x_1, y_1 \rangle_1 \dots \langle x_n,y_n \rangle_n, \tag{1.4}$$

for every $x,y \varepsilon H_1 \otimes \dots \otimes H_n$.

DEFINITION 1.4. For the operators $A_j \varepsilon L(H_j)$, the tensor product operators, $A_j^{\otimes}$ $(1 < j < n)$ is defined

$$A_j^{\otimes} = I_1 \otimes \dots \otimes I_{j-1} \otimes A_j \otimes I_{j+1} \otimes \dots \otimes I_n, \tag{1.5}$$

where I_j is the identity operator on H_j.

Clearly, $A^{\otimes} = (A_1^{\otimes},\dots, A_n^{\otimes})$ is a commuting system of operators on $\hat{H}$.

2. A MAIN RESULT ON THE JOINT NUMERICAL STATUS.

We now prove the following result on the joint numerical status:

THEOREM 2.1. Let $A^{\otimes} = (A_1^{\otimes},\dots, A_n^{\otimes}) \varepsilon C^* (L(\hat{H}))$. Then

$$S(A_1^{\otimes},\dots,A_n^{\otimes}) = S(A_1) \times\dots\times S(A_n). \tag{2.1}$$

PROOF. In order to prove the inclusion

$$S(A_1^{\otimes},\dots,A_n^{\otimes}) \subseteq S(A_1) \times\dots\times S(A_n), \tag{2.2}$$

we first prove that $S(A_j^{\otimes}) = S(A_j)$ $(j = 1,\dots,n)$. Since this follows by induction from the case of $n = 2$, we first show that $S(A_1 \otimes I_2) = S(A_1)$.

If $\lambda \in S(A_1)$, then there exists a state f for A_1 on $C^*(L(H_1))$ such that $f(A_1) = \lambda$. Let g be any state on $C^*(L(H_2))$. Then

$$(f \otimes g)\ (A_1 \otimes I_2) = f(A_1) \otimes g(I_2) = f(A_1). \tag{2.3}$$

This implies that $f \otimes g$ is a state for $A_1 \otimes I_2$ on $C^*(L(H_1 \hat{\otimes} H_2))$, and $\lambda \in S(A_1 \otimes I_2)$.

Thus

$$S(A_1) \subseteq S(A_1 \otimes I_2). \tag{2.4}$$

Conversely, $\lambda \in S(A_1 \otimes I_2)$, then there exists a state h for $A_1 \otimes I_2$ such that $h(A_1 \otimes I_2) = \lambda$.

We next define a functional f for each $A \in C^*(L(H_1))$ such that $f(A) = h(A \otimes I_2)$. Then $f(I_1) = h(I_1 \otimes I_2) = 1$,

$$||f|| = \sup_{||A|| \leq 1} |f(A)| = \sup_{||A \otimes I_2|| \leq 1} |h(A \otimes I_2)|$$

$$\leq ||h|| = 1, \text{ and } ||f|| = 1.$$

Hence

$$\lambda = h(A_1 \otimes I_2) = f(A_1) \in S(A_1).$$

This implies that $S(A_1 \otimes I_2) \subseteq S(A_1)$. This completes the proof of this part.

Similarly, we can prove that $S(I_3 \otimes A_1) = S(A_1)$. This result combined with inclusion (1.3) establishes the theorem one way.

To prove converse part of the Theorem, let $\lambda = (\lambda_1,\ldots,\lambda_n)$ be an element of $S(A_1) \times\ldots\times S(A_n)$. Then there exists some state f_j for A_j such that $\lambda_j = f_j(A_j)$ for all j, $1 \leq j \leq n$. Now, if we set

$$f^{\otimes} = f_1 \otimes \ldots \otimes f_{j-1} \otimes f_j \otimes f_{j+1} \otimes \ldots \otimes f_n,$$

then $f^{\otimes}$ is a state on $C^*(L(H_1 \hat{\otimes} \ldots \hat{\otimes} H_n))$, and $\lambda_j = f^{\otimes}(A_j^{\otimes}) = f_j(A_j)$ for all j, $1 \leq j \leq n$. This implies that $\lambda \in S(A_1^{\otimes},\ldots,A_n^{\otimes})$, and

$$S(A_1) \times\ldots\times S(A_n) \subseteq S(A_1,\ldots,A_n).$$

This completes the proof.

REMARK. For $A_1,\ldots,A_n$ commuting normal operators, we find the connection

$$S(A_1^{\otimes},\ldots,A_n^{\otimes}) = W(A_1^{\otimes},\ldots,A_n^{\otimes})^{-} = co(\sigma_{\pi}(A_1^{\otimes},\ldots,A_n^{\otimes})),$$

where the bar denotes the closure, and $co(\sigma_{\pi}(.))$ denotes the convex hull of the joint aproximate point spectrum [1].

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