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ON THE WEAK LAW OF LARGE NUMBERS FOR NORMED WEIGHTED SUMS OF I.I.D. RANDOM VARIABLES

ANDRÉ ADLER

Department of Mathematics
Illinois Institute of Technology
Chicago, Illinois 60616 U.S.A.

ANDREW ROSALSKY

Department of Statistics
University of Florida
Gainesville, Florida 32611 U.S.A.

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ABSTRACT. For weighted sums $\sum_{j=1}^{n} a_j Y_j$ of independent and identically distributed random variables $\{Y_n, n \geq 1\}$, a general weak law of large numbers of the form $\left(\sum_{j=1}^{n} a_j Y_j - \nu_n\right) / b_n \xrightarrow{P} 0$ is established where $\{\nu_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ are suitable constants. The hypotheses involve both the behavior of the tail of the distribution of $|Y_1|$ and the growth behaviors of the constants $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$. Moreover, a weak law is proved for weighted sums $\sum_{j=1}^{T_n} a_j Y_j$ indexed by random variables $\{T_n, n \geq 1\}$. An example is presented wherein the weak law holds but the strong law fails thereby generalizing a classical example.

KEY WORDS AND PHRASES. Weighted sums of independent and identically distributed random variables, weak law of large numbers, convergence in probability, random indices, strong law of large numbers, almost certain convergence.

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1. INTRODUCTION.

Let $\{Y, Y_n, n \geq 1\}$ be independent and identically distributed (i.i.d.) random variables defined on a probability space $(\Omega, \mathcal{F}, P)$, and let $\{a_n, n \geq 1\}$, $\{\nu_n, n \geq 1\}$, and $\{b_n, n \geq 1\}$ be constants with $a_n \neq 0$, $b_n > 0$, $n \geq 1$. Then $\{a_n Y_n, n \geq 1\}$ is said to obey the general weak law of large numbers (WLLN) with centering constants $\{\nu_n, n \geq 1\}$ and norming constants $\{b_n, n \geq 1\}$ if the normed and centered weighted sum $\left(\sum_{j=1}^{n} a_j Y_j - \nu_n\right) / b_n$ has the weak limiting behavior

$$\frac{\sum_{j=1}^{n} a_j Y_j - \nu_n}{b_n} \xrightarrow{P} 0 \tag{1.1}$$

where $\xrightarrow{P}$ denotes convergence in probability. Herein, the main result, Theorem 1, furnishes conditions on $\{a_n, n \geq 1\}$, $\{b_n, n \geq 1\}$, and the distribution of Y which ensure that $\{a_n Y_n, n \geq 1\}$ obeys the WLLN (1.1) for suitable $\{\nu_n, n \geq 1\}$. It is not assumed that Y is integrable. Of course, the well-known degenerate convergence criterion (see, e.g., Loève [1, p. 329]) solves, in theory, the WLLN problem. The advantage of employing Theorem 1 lies in the fact that, in practice, its conditions (2.1), (2.2), and (2.3) are simpler and more easily verifiable than the hypotheses of the degenerate convergence criterion. Jamison et al. [2] had investigated the WLLN problem in the special case where $a_n > 0$, $b_n = \sum_{j=1}^{n} a_j$, $n \geq 1$, and $\max_{1 \leq j \leq n} a_j = o(b_n)$.

Conditions for $\{a_nY_n, n\geq 1\}$ to obey the general strong law of large numbers (SLLN)

$$\frac{\sum_{j=1}^{n} a_jY_j - \nu_n}{b_n} \to 0 \text{ almost certainly (a.c.)}$$

had been obtained by Adler and Rosalsky [3,4]. In Section 5, an example illustrating Theorem 1 is presented and the corresponding SLLN is shown to fail.

The WLLN problem is studied in Theorem 2 in the more general context of random indices. More specifically, let $\{T_n, n\geq 1\}$ be positive integer-valued random variables and let $1\leq \alpha_n \to \infty$ be constants such that $P\{T_n/\alpha_n > \lambda\} = o(1)$ for some $\lambda > 0$. Theorem 2 provides conditions for

$$\frac{\sum_{j=1}^{T_n} a_jY_j - \nu_{[\alpha_n]}}{b_{[\alpha_n]}} \xrightarrow{P} 0,$$

where the symbol $[x]$ denotes the greatest integer in x.

As will become apparent, Theorem 2 of Klass and Teicher [5] and Theorem 5.2.6 of Chow and Teicher [6, p. 131] provided, respectively, the motivation for Theorems 1 and 2 herein. Moreover, our Theorems 1 and 2 are proved using an approach similar to that of the earlier counterparts.

Some remarks about notation are in order. Throughout, a sequence $\{c_n, n\geq 1\}$ is defined by $c_n = b_n/|a_n|$, $n\geq 1$, and the symbol C denotes a generic constant $(0<C<\infty)$ which is not necessarily the same one in each appearance. The symbols $u_n\uparrow$ or $u_n\downarrow$ are used to indicate that the given numerical sequence $\{u_n, n\geq 1\}$ is monotone increasing or monotone decreasing, respectively.

2. A PRELIMINARY LEMMA.

The key lemma in establishing Theorems 1 and 2 will now be stated and proved. It should be noted that the conditions (2.2) and (2.3) are automatically satisfied for the standard assignment of $a_n=1$, $b_n=n$, $n\geq 1$.

LEMMA. If

$$nP\{|Y|>c_n\} = o(1) \tag{2.1}$$

and either

$$c_n\uparrow,\ \frac{c_n}{n}\downarrow,\ \sum_{j=1}^{n} a_j^2 = o\left(b_n^2\right),\ \text{and}\ \sum_{j=1}^{n}\left(\frac{c_j}{j}\right)^2 = O\left(\frac{b_n^2}{\sum_{j=1}^{n} a_j^2}\right) \tag{2.2}$$

or

$$\frac{c_n}{n}\uparrow \text{ and } \sum_{j=1}^{n} a_j^2 = O\left(na_n^2\right), \tag{2.3}$$

then

$$\sum_{j=1}^{n} a_j^2 EY^2 I(|Y|\leq c_n) = o\left(b_n^2\right).$$

PROOF. Note at the outset that $c_n\uparrow$ under either (2.2) or (2.3) and that (2.3) ensures

$$\sum_{j=1}^{n} a_j^2 = o\left(b_n^2\right). \tag{2.4}$$

Thus, (2.4) holds under either (2.2) or (2.3). Let $c_0 = 0$ and $d_n = c_n/n$, $n\geq 1$. Define an array $\{B_{nk}, 0\leq k\leq n, n\geq 1\}$ by

$$B_{nk} = \begin{cases} \left(\frac{1}{b_n^2}\sum_{j=1}^n a_j^2\right)\left(\frac{c_{k+1}^2 - c_k^2}{k}\right) & \text{for } 1 \le k \le n-1,\ n \ge 2 \\ 0 \quad \text{for } k = 0,\ n,\ n \ge 1. \end{cases}$$

It will now be shown that $\{B_{nk}, 0 \le k \le n, n \ge 1\}$ is a Toeplitz array, that is,

$$\sum_{k=0}^n |B_{nk}| = O(1) \tag{2.5}$$

and

$$B_{nk} \to 0 \text{ as } n \to \infty \text{ for all fixed } k \ge 0. \tag{2.6}$$

Clearly (2.4) entails (2.6). To verify (2.5), note that $B_{nk} \ge 0$, $0 \le k \le n$, $n \ge 1$, since $c_n \uparrow$. Now under (2.2), for all $n \ge 2$,

$$\sum_{k=0}^n B_{nk} = \left(\frac{1}{b_n^2}\sum_{j=1}^n a_j^2\right)\left(\sum_{k=1}^{n-1} \frac{(k+1)^2 d_{k+1}^2 - k^2 d_k^2}{k}\right)$$

$$\le \left(\frac{1}{b_n^2}\sum_{j=1}^n a_j^2\right)\left(\sum_{k=1}^{n-1} \frac{\left((k+1)^2 - k^2\right) d_k^2}{k}\right) \quad (\text{since } d_n \downarrow)$$

$$\le \left(\frac{3}{b_n^2}\sum_{j=1}^n a_j^2\right)\left(\sum_{k=1}^{n-1} d_k^2\right) = O(1)$$

and so (2.5) holds. On the other hand, under (2.3),

$$d_n \uparrow \text{ and } \sum_{j=1}^n a_j^2 \le C n a_n^2,\ n \ge 1.$$

Then for all $n \ge 1$,

$$\frac{\sum_{j=1}^n a_j^2}{b_n^2} \le \frac{Cn}{c_n^2} = \frac{C}{n d_n^2}.$$

Thus for all $n \ge 2$,

$$\sum_{k=0}^n B_{nk} \le \left(\frac{C}{n d_n^2}\right)\left(\sum_{k=1}^{n-1} \frac{(k+1)^2 d_{k+1}^2 - k^2 d_k^2}{k}\right)$$

$$\le \left(\frac{C}{n d_n^2}\right)\left(\sum_{k=1}^{n-1}\left((k+3) d_{k+1}^2 - k d_k^2\right)\right)$$

$$= \left(\frac{C}{n d_n^2}\right)\left(\sum_{k=1}^{n-1}\left((k+1) d_{k+1}^2 - k d_k^2\right)\right) + \left(\frac{2C}{n d_n^2}\right)\left(\sum_{k=1}^{n-1} d_{k+1}^2\right)$$

$$\le \frac{C n d_n^2}{n d_n^2} + \frac{2C(n-1) d_n^2}{n d_n^2} \quad (\text{since } d_n \uparrow)$$

$$= O(1)$$

and again (2.5) holds thereby proving that $\{B_{nk}, 0 \le k \le n, n \ge 1\}$ is a Toeplitz array.

Then by (2.1) and the Toeplitz lemma (see, e.g., Knopp [7, p. 74] or Loève [1, p. 250]),

$$\sum_{k=0}^{n} B_{nk} kP\{|Y|>c_k\} = o(1). \tag{2.7}$$

Next, note that

$$\frac{1}{b_n^2}\sum_{j=1}^{n} a_j^2 EY^2 I(|Y|\le c_n)$$

$$= \frac{1}{b_n^2}\sum_{j=1}^{n} a_j^2 \sum_{k=1}^{n} EY^2 I(c_{k-1}<|Y|\le c_k)$$

$$\le \frac{1}{b_n^2}\sum_{j=1}^{n} a_j^2 \sum_{k=1}^{n} c_k^2 P\{c_{k-1}<|Y|\le c_k\}$$

$$= \frac{1}{b_n^2}\sum_{j=1}^{n} a_j^2 \sum_{k=1}^{n} c_k^2\Big(P\{|Y|>c_{k-1}\} - P\{|Y|>c_k\}\Big)$$

$$= \frac{1}{b_n^2}\sum_{j=1}^{n} a_j^2\Big(c_1^2 P\{|Y|>0\} - c_n^2 P\{|Y|>c_n\} + \sum_{k=1}^{n-1}(c_{k+1}^2-c_k^2)P\{|Y|>c_k\}\Big)$$

(by the Abel "summation by parts" lemma)

$$\le \frac{1}{b_n^2}\sum_{j=1}^{n} a_j^2 \sum_{k=1}^{n-1}\left(\frac{c_{k+1}^2-c_k^2}{k}\right)kP\{|Y|>c_k\} + o(1) \quad \Big(\text{by } (2.4)\Big)$$

$$= \sum_{k=0}^{n} B_{nk} kP\{|Y|>c_k\} + o(1)$$

$$= o(1) \quad \Big(\text{by } (2.7)\Big)$$

thereby proving the Lemma. □

3. THE MAIN RESULT.

With the preliminaries accounted for, Theorem 1 may be stated and proved. As was noted in the proof of the Lemma, the hypotheses to Theorem 1 entail (2.4) and so necessarily $b_n \to \infty$. However, it is not assumed that $\{b_n, n\ge 1\}$ is monotone. (In most SLLN results, monotonicity of $\{b_n, n\ge 1\}$ is assumed.)

THEOREM 1. Let $\{Y, Y_n, n\ge 1\}$ be i.i.d. random variables and let $\{a_n, n\ge 1\}$ and $\{b_n, n\ge 1\}$ be constants satisfying $a_n \ne 0$, $b_n>0$, $n\ge 1$, and either (2.2) or (2.3). If (2.1) holds, then the WLLN

$$\frac{\sum_{j=1}^{n} a_j\Big(Y_j - EYI(|Y|\le c_n)\Big)}{b_n} \xrightarrow{P} 0 \tag{3.1}$$

obtains.

PROOF. Define $Y_{nj} = Y_j I(|Y_j|\le c_n)$, $1\le j\le n$, $n\ge 1$. For arbitrary $\epsilon>0$,

$$P\left\{\frac{\left|\sum_{j=1}^{n} a_j\Big(Y_j - Y_{nj}\Big)\right|}{b_n} > \epsilon\right\} \le P\left\{\bigcup_{j=1}^{n}\Big[Y_j \ne Y_{nj}\Big]\right\} \le nP\{|Y|>c_n\} = o(1) \quad \Big(\text{by } (2.1)\Big),$$

whence

$$\frac{\sum_{j=1}^{n} a_j\left(Y_j - Y_{nj}\right)}{b_n} \xrightarrow{P} 0. \tag{3.2}$$

Also,

$$\frac{\sum_{j=1}^{n} a_j\left(Y_{nj} - EY_{nj}\right)}{b_n} \xrightarrow{P} 0 \tag{3.3}$$

since for arbitrary $\epsilon > 0$,

$$P\left\{\frac{\left|\sum_{j=1}^{n} a_j\left(Y_{nj} - EY_{nj}\right)\right|}{b_n} > \epsilon\right\} \leq \frac{1}{\epsilon^2 b_n^2} \sum_{j=1}^{n} a_j^2 EY^2 I(|Y| \leq c_n) = o(1)$$

by the Lemma. The conclusion (3.1) follows directly from (3.2) and (3.3). □

REMARKS. (i) Apropos of the condition (2.2), if c_n/n is slowly varying at infinity and $\sum_{j=1}^{n} a_j^2 = O\left(na_n^2\right)$, then

$$\sum_{j=1}^{n} a_j^2 = o\left(b_n^2\right) \quad \text{and} \quad \sum_{j=1}^{n}\left(\frac{c_j}{j}\right)^2 = O\left(\frac{b_n^2}{\sum_{j=1}^{n} a_j^2}\right).$$

PROOF. Note that

$$\frac{\sum_{j=1}^{n} a_j^2}{b_n^2} \leq \frac{Cna_n^2}{b_n^2} = \frac{C(n/c_n)^2}{n} = o(1)$$

by slow variation (see, e.g., Seneta [8, p. 18]). Then slow variation yields (see, e.g., Feller [9, p. 281])

$$\sum_{j=1}^{n}\left(\frac{c_j}{j}\right)^2 \sim n\left(\frac{c_n}{n}\right)^2 = \frac{b_n^2}{na_n^2} = O\left(\frac{b_n^2}{\sum_{j=1}^{n} a_j^2}\right). \qquad \square$$

(ii) Adler [10] proved a partial converse of Theorem 1.

(iii) In the spirit of Klass and Teicher [5], Adler [11] has employed Theorem 1 to obtain a generalized one-sided law of the iterated logarithm (LIL) for weighted sums of i.i.d. random variables barely with or without finite mean thereby generalizing some of the work of [5]. (Corollary 1 below had been obtained by Klass and Teicher [5] and they used it in their investigation of the LIL for i.i.d. asymmetric random variables.) To be somewhat more specific, Adler [11] employed the WLLN (3.1) to obtain the a.c. limiting value of some (nonrandom) subsequence of $\sum_{j=1}^{n} a_j Y_j / b_n$ thereby yielding an upper bound for the a.c. value of $\liminf_{n\to\infty} \sum_{j=1}^{n} a_j Y_j / b_n$.

The ensuing Corollary 1 is a WLLN analogue of Feller's [12] famous generalization of the Marcinkiewicz-Zygmund SLLN.

COROLLARY 1 (Klass and Teicher [5]). Let $\{Y, Y_n, n \geq 1\}$ be i.i.d. random variables and let $\{b_n, n \geq 1\}$ be positive constants such that either $b_n/n\uparrow$ or

$$b_n\uparrow, \ \frac{b_n}{n}\downarrow, \ \frac{b_n}{\sqrt{n}} \to \infty, \ \text{and} \ \sum_{j=1}^{n}\left(\frac{b_j}{j}\right)^2 = O\left(\frac{b_n^2}{n}\right). \tag{3.4}$$

Then

$$\frac{\sum_{j=1}^{n} Y_j - nEYI(|Y| \leq b_n)}{b_n} \xrightarrow{P} 0 \ \text{ iff } \ nP\{|Y| > b_n\} = o(1).$$

PROOF. Sufficiency follows directly from Theorem 1 whereas necessity follows from the degenerate convergence criterion noting that the family $\left\{\left(Y_j - EYI(|Y|\leq b_n)\right)/b_n,\ 1\leq j\leq n,\ n\geq 1\right\}$ is uniformly asymptotically negligible. □

REMARK. In the Klass-Teicher [5] version of Corollary 1, the second condition of the assumption (3.4) appears in the stronger form $b_n/n\downarrow 0$.

The next corollary is an immediate consequence of Corollary 1 and is the classical WLLN attributed to Feller by Chow and Teicher [6, p. 128].

COROLLARY 2. If $\{Y, Y_n, n\geq 1\}$ are i.i.d. random variables, then

$$\frac{\sum_{j=1}^{n} Y_j - \nu_n}{n} \xrightarrow{P} 0$$

for some choice of centering constants $\{\nu_n, n\geq 1\}$ iff

$$nP\{|Y|>n\} = o(1).$$

In such a case, $\nu_n/n = EYI(|Y|\leq n) + o(1)$.

The next corollary removes the indicator function from the expression in (3.1).

COROLLARY 3. Let $\{Y, Y_n, n\geq 1\}$ be i.i.d. L_1 random variables and let $\{a_n, n\geq 1\}$ and $\{b_n, n\geq 1\}$ be constants satisfying $a_n\neq 0$, $b_n>0$, $n\geq 1$, and either (2.2) or (2.3). If (2.1) holds and $M \equiv \lim_{n\to\infty}\sum_{j=1}^{n} a_j/b_n$ exists and is finite, then

$$\frac{\sum_{j=1}^{n} a_j Y_j}{b_n} \xrightarrow{P} M(EY).$$

PROOF. First observe that (3.1) obtains by Theorem 1. Now $\lim_{n\to\infty} c_n = \infty$ since $a_n^2 = o\left(b_n^2\right)$ by (2.4). Then by the Lebesgue dominated convergence theorem, $EYI(|Y|\leq c_n)\to EY$, whence

$$\frac{\sum_{j=1}^{n} a_j EYI(|Y|\leq c_n)}{b_n} \to M(EY)$$

which when combined with (3.1) yields the conclusion. □

4. A WLLN WITH RANDOM INDICES.

In this section, Theorem 1 is extended to the case of random indices $\{T_n, n\geq 1\}$. No assumptions are made regarding the joint distributions of $\{T_n, n\geq 1\}$ whose marginal distributions are constrained solely by (4.1). Moreover, it is not assumed that the sequences $\{T_n, n\geq 1\}$ and $\{Y_n, n\geq 1\}$ are independent of each other. It should be noted that the condition (4.1) is considerably weaker than $T_n/\alpha_n \xrightarrow{P} c$ for some constant $c\in[0,\infty)$.

THEOREM 2. Let $\{Y, Y_n, n\geq 1\}$, $\{a_n, n\geq 1\}$, and $\{b_n, n\geq 1\}$ satisfy the hypotheses of Theorem 1 and let $\{T_n, n\geq 1\}$ be positive integer-valued random variables and $1\leq \alpha_n\to\infty$ be constants such that for some $\lambda>0$

$$P\left\{\frac{T_n}{\alpha_n} > \lambda\right\} = o(1) \tag{4.1}$$

and

$$b_{[\lambda\alpha_n]} = O\left(b_{[\alpha_n]}\right) \quad \text{if } \lambda>1 \tag{4.2}$$

hold. Then

$$\frac{\sum_{j=1}^{T_n} a_j\left(Y_j - EYI\left(|Y| \le c_{[\alpha n]}\right)\right)}{b_{[\alpha n]}} \xrightarrow{P} 0.$$

PROOF. Let $Y_{nj} = Y_j I\left(|Y_j| \le c_{[\alpha n]}\right)$, $j \ge 1$, $n \ge 1$. Firstly, it will be verified that

$$\frac{\sum_{j=1}^{T_n} a_j\left(Y_j - Y_{nj}\right)}{b_{[\alpha n]}} \xrightarrow{P} 0. \tag{4.3}$$

For arbitrary $\epsilon > 0$ and all large n,

$$\begin{aligned}
&P\left\{\frac{\left|\sum_{j=1}^{T_n} a_j\left(Y_j - Y_{nj}\right)\right|}{b_{[\alpha n]}} > \epsilon\right\} \\
&\le P\left\{\sum_{j=1}^{T_n} a_j Y_j \ne \sum_{j=1}^{T_n} a_j Y_{nj}\right\} \\
&\le P\left\{\left[\sum_{j=1}^{T_n} a_j Y_j \ne \sum_{j=1}^{T_n} a_j Y_{nj}\right]\left[T_n \le \lambda\alpha_n\right]\right\} + P\left\{T_n > \lambda\alpha_n\right\} \\
&\le P\left\{\bigcup_{j=1}^{[\lambda\alpha_n]}\left[|Y_j| > c_{[\alpha_n]}\right]\right\} + o(1) \quad \left(\text{by } (4.1)\right) \\
&\le [\lambda\alpha_n] P\left\{|Y| > c_{[\alpha_n]}\right\} + o(1) \\
&= \left(1 + o(1)\right)\lambda[\alpha n] P\left\{|Y| > c_{[\alpha_n]}\right\} + o(1) \\
&= o(1) \quad \left(\text{by } (2.1)\right)
\end{aligned}$$

thereby establishing (4.3).

Thus, to complete the proof, it only needs to be demonstrated that

$$\frac{\sum_{j=1}^{T_n} a_j\left(Y_{nj} - EY_{nj}\right)}{b_{[\alpha n]}} \xrightarrow{P} 0. \tag{4.4}$$

To this end, for arbitrary $\epsilon > 0$ and all large n,

$$\begin{aligned}
&P\left\{\frac{\left|\sum_{j=1}^{T_n} a_j\left(Y_{nj} - EY_{nj}\right)\right|}{b_{[\alpha n]}} > \epsilon\right\} \\
&\le P\left\{\left[\frac{\left|\sum_{j=1}^{T_n} a_j\left(Y_{nj} - EY_{nj}\right)\right|}{b_{[\alpha n]}} > \epsilon\right]\left[T_n \le \lambda\alpha_n\right]\right\} + P\left\{T_n > \lambda\alpha_n\right\}
\end{aligned}$$

$$\le P\left\{\bigcup_{k=1}^{[\lambda\alpha_n]}\left[\left|\sum_{j=1}^{k} a_j\left(Y_{nj} - EY_{nj}\right)\right| > \epsilon b_{[\alpha_n]}\right]\right\} + o(1) \quad \left(\text{by (4.1)}\right)$$

$$= P\left\{\max_{1\le k\le[\lambda\alpha_n]}\left|\sum_{j=1}^{k} a_j\left(Y_{nj} - EY_{nj}\right)\right| > \epsilon b_{[\alpha_n]}\right\} + o(1)$$

$$\le \frac{1}{\epsilon^2 b^2_{[\alpha_n]}}\sum_{j=1}^{[\lambda\alpha_n]} \mathrm{Var}\left(a_j Y_{nj}\right) + o(1) \quad \text{(by the Kolmogorov inequality)}$$

$$\le \frac{1}{\epsilon^2 b^2_{[\alpha_n]}}\sum_{j=1}^{[\lambda\alpha_n]} a_j^2 EY^2 I\left(|Y|\le c_{[\alpha_n]}\right) + o(1)$$

$$\le \begin{cases} \dfrac{1}{\epsilon^2 b^2_{[\alpha_n]}}\displaystyle\sum_{j=1}^{[\alpha_n]} a_j^2 EY^2 I\left(|Y|\le c_{[\alpha_n]}\right) + o(1) & \text{if } 0<\lambda\le 1 \\ \dfrac{C}{b^2_{[\lambda\alpha_n]}}\displaystyle\sum_{j=1}^{[\lambda\alpha_n]} a_j^2 EY^2 I\left(|Y|\le c_{[\lambda\alpha_n]}\right) + o(1) & \text{if } \lambda>1 \end{cases} \quad \left(\text{by (4.2) and } c_n\uparrow\right)$$

$$= o(1) \quad \text{(by the Lemma)}$$

thereby establishing (4.4) and Theorem 2. □

REMARKS. (i) The referee to this paper so kindly supplied the following example which shows that Theorem 2 can fail if the norming sequence $\left\{b_{[\alpha_n]}, n\ge 1\right\}$ is replaced by $\{T_n, n\ge 1\}$. Let $\{Y, Y_n, n\ge 1\}$ be i.i.d. Cauchy random variables and let

$$a_n=1,\ b_n=n^{1+\epsilon},\ T_n=n,\ \alpha_n=n,\ n\ge 1$$

where $\epsilon>0$. Then (2.1) and (2.3) hold and trivially $T_n/\alpha_n \xrightarrow{P} 1$ and hence the conclusion to Theorem 2 obtains, but

$$\frac{\sum_{j=1}^{T_n} a_j\left(Y_j - EYI\left(|Y|\le c_{[\alpha_n]}\right)\right)}{T_n} = \frac{\sum_{j=1}^{n} Y_j}{n} \overset{P}{\not\to} 0.$$

(ii) The referee also suggested that the authors look into the question as to whether in Theorem 2 the norming sequence can be taken to be $\left\{b_{T_n}, n\ge 1\right\}$. The ensuing corollary provides conditions for the answer to be affirmative. It should be noted that the pair of conditions (4.1) and (4.5) is equivalent to the single condition

$$P\left\{\lambda' \le \frac{T_n}{\alpha_n} \le \lambda\right\} \to 1 \text{ for some } \lambda\ge\lambda'>0$$

which is clearly weaker than $T_n/\alpha_n \xrightarrow{P} c$ for some constant $0<c<\infty$.

COROLLARY 4. Let $\{Y, Y_n, n\ge 1\}$, $\{a_n, n\ge 1\}$, $\{b_n, n\ge 1\}$, and $\{\alpha_n, n\ge 1\}$ satisfy the hypotheses of Theorem 2 and suppose, additionally, that $b_n\uparrow$ and for some $\lambda'>0$ that

$$P\left\{\frac{T_n}{\alpha_n} < \lambda'\right\} = o(1) \tag{4.5}$$

and

$$b_{[\alpha_n]} = O\left(b_{[\lambda'\alpha_n]}\right) \text{ if } \lambda'<1 \tag{4.6}$$

hold. Then

$$\frac{\sum_{j=1}^{T_n} a_j\left(Y_j - EYI\left(|Y|\leq c_{[\alpha_n]}\right)\right)}{b_{T_n}} \xrightarrow{P} 0.$$

PROOF. In view of Theorem 2, it suffices to show that $b_{[\alpha_n]}/b_{T_n}$ is bounded in probability, that is, for all $\epsilon>0$, there exists a constant $C<\infty$ and an integer N such that for all $n\geq N$

$$P\left\{\frac{b_{[\alpha_n]}}{b_{T_n}}>C\right\} \leq \epsilon. \tag{4.7}$$

To this end, let $\epsilon>0$. If $\lambda'\geq 1$, then letting C=1, the monotonicity of $\{b_n, n\geq 1\}$ guarantees that

$$b_{[\alpha_n]} \leq Cb_{[\lambda'\alpha_n]},\ n\geq 1 \tag{4.8}$$

whereas if $\lambda'<1$, then (4.6) ensures (4.8) for some constant $C<\infty$. Thus, (4.8) holds in either case. Then for all large n,

$$\begin{aligned} &P\left\{\frac{b_{[\alpha_n]}}{b_{T_n}}>C\right\} \\ &\leq P\left\{\left[b_{[\alpha_n]}>Cb_{T_n}\right]\left[T_n\geq[\lambda'\alpha_n]\right]\right\} + P\left\{T_n<[\lambda'\alpha_n]\right\} \\ &\leq P\left\{b_{[\alpha_n]}>Cb_{[\lambda'\alpha_n]}\right\} + \epsilon \quad \left(\text{by } b_n\uparrow \text{ and (4.5)}\right) \\ &= \epsilon \quad \left(\text{by (4.8)}\right) \end{aligned}$$

thereby establishing (4.7) and Corollary 4. □

(iii) The ensuing example shows that, in general, Theorem 2 can fail if the norming sequence $\left\{b_{[\alpha_n]}, n\geq 1\right\}$ is replaced by $\left\{b_{T_n}, n\geq 1\right\}$. Let $\{Y, Y_n, n\geq 1\}$ be i.i.d. random variables with Y having probability density function

$$f(y) = \frac{C}{y^2\log y} I_{[e,\infty)}(y),\ -\infty<y<\infty$$

where C is a constant and let

$$a_n=1,\ b_n=n,\ T_n=[\sqrt{n}],\ \alpha_n=n,\ n\geq 1.$$

Now for all $n\geq 3$, employing Theorem 1 of Feller [9, p. 281],

$$nP\{|Y|>n\} = nC\int_n^\infty \frac{1}{y^2\log y}\,dy = \frac{\left(1+o(1)\right)C}{\log n} = o(1).$$

All of the hypotheses to Theorem 2 are satisfied and hence the conclusion to Theorem 2 obtains. Assume, however, that

$$\frac{\sum_{j=1}^{T_n} a_j\left(Y_j - EYI\left(|Y|\le c_{[\alpha_n]}\right)\right)}{b_{T_n}} = \frac{\sum_{j=1}^{[\sqrt{n}]}\left(Y_j - EYI\left(|Y|\le n\right)\right)}{[\sqrt{n}]} \xrightarrow{P} 0 \tag{4.9}$$

prevails. Then

$$\frac{\sum_{j=1}^{n} Y_j}{n} - EYI\left(|Y|\le n^2\right) = \frac{\sum_{j=1}^{n}\left(Y_j - EYI\left(|Y|\le n^2\right)\right)}{n} \xrightarrow{P} 0.$$

But by Corollary 2,

$$\frac{\sum_{j=1}^{n} Y_j}{n} - EYI(|Y|\le n) \xrightarrow{P} 0.$$

whence via subtraction $EYI\left(n<|Y|\le n^2\right) = o(1)$. But for $n\ge 3$,

$$EYI\left(n<|Y|\le n^2\right) = \int_n^{n^2} \frac{C}{y \log y}\, dy = C\left(\log\log n^2 - \log\log n\right) = C \log 2,$$

a contradiction. Thus, (4.9) must fail.

The last corollary of this section, Corollary 5, is a random indice version of the sufficiency half of Corollary 2, and it is Theorem 5.2.6 of Chow and Teicher [6, p. 131]. Corollary 5 follows immediately from Corollary 4 by taking $a_n=1$, $b_n=n$, $\alpha_n=n$, $n\ge 1$.

COROLLARY 5. Let $\{Y, Y_n, n\ge 1\}$ be i.i.d. random variables such that $nP\{|Y|>n\} = o(1)$ and let $\{T_n, n\ge 1\}$ be positive integer-valued random variables such that

$$\frac{T_n}{n} \xrightarrow{P} c \text{ for some constant } 0<c<\infty.$$

Then

$$\frac{\sum_{j=1}^{T_n} Y_j}{T_n} - EYI(|Y|\le n) \xrightarrow{P} 0.$$

5. AN INTERESTING EXAMPLE.

In this last section, a generalization of a classical example is presented. A sequence of weighted i.i.d. random variables $\{a_n Y_n, n\ge 1\}$ is shown, via Theorem 1, to obey a WLLN. On the other hand, the corresponding SLLN is shown to fail. It should be noted that $E|Y| = \infty$. The classical example is the special case $\delta=1$ and $a_n \equiv 1$.

EXAMPLE. Let $\{Y, Y_n, n\ge 1\}$ be i.i.d. random variables with Y having probability density function

$$f(y) = \frac{C_\delta}{y^2(\log |y|)^\delta} I_{(-\infty,-e]\cup[e,\infty)}(y), \quad -\infty<y<\infty$$

where $0<\delta\le 1$ and C_δ is a constant. Then for every sequence of constants $\{a_n, n\ge 1\}$ with $0<|a_n|\uparrow$,

$$\frac{\sum_{j=1}^{n} a_j Y_j}{n|a_n|} \xrightarrow{P} 0, \tag{5.1}$$

but

$$\limsup_{n\to\infty} \frac{\sum_{j=1}^{n} a_jY_j}{n|a_n|} = -\liminf_{n\to\infty} \frac{\sum_{j=1}^{n} a_jY_j}{n|a_n|} = \infty \text{ a.c.} \tag{5.2}$$

and, consequently, for any constant $c\in(-\infty,\infty)$

$$P\left\{\lim_{n\to\infty} \frac{\sum_{j=1}^{n} a_jY_j}{n|a_n|} = c\right\} = 0.$$

PROOF. Set $b_n=n|a_n|$, $n\geq 1$. Then $c_n=n$, $n\geq 1$, and both (2.2) and (2.3) hold. Now for all $n\geq 3$, employing Theorem 1 of Feller [9, p. 281],

$$nP\{|Y|>n\} = 2nC_\delta \int_n^\infty \frac{1}{y^2(\log y)^\delta}\,dy = \frac{\big(1+o(1)\big)2C_\delta}{(\log n)^\delta} = o(1),$$

and so (5.1) follows from Theorem 1 since $EYI(|Y|\leq n) = 0$, $n\geq 1$.

Next, for arbitrary $0<M<\infty$, $E\frac{|Y|}{M} = \infty$ ensures that

$$\sum_{n=1}^{\infty} P\left\{\frac{|Y_n|}{M} > n\right\} = \infty,$$

whence by the Borel-Cantelli lemma

$$P\left\{\limsup_{n\to\infty} \frac{|Y_n|}{n} \geq M\right\} \geq P\left\{\frac{|Y_n|}{n} > M \text{ i.o.}(n)\right\} = 1.$$

Since M is arbitrary,

$$\infty = \limsup_{n\to\infty} \frac{|Y_n|}{n} = \limsup_{n\to\infty} \frac{\left|\sum_{j=1}^{n} a_jY_j - \sum_{j=1}^{n-1} a_jY_j\right|}{n|a_n|}$$

$$\leq \limsup_{n\to\infty} \frac{\left|\sum_{j=1}^{n} a_jY_j\right|}{n|a_n|} + \limsup_{n\to\infty} \frac{\left|\sum_{j=1}^{n-1} a_jY_j\right|}{(n-1)|a_{n-1}|} \text{ a.c.},$$

and so

$$\limsup_{n\to\infty} \frac{\left|\sum_{j=1}^{n} a_jY_j\right|}{n|a_n|} = \infty \text{ a.c.}$$

implying (5.2) via symmetry and the Kolmogorov 0–1 law. □

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REFERENCES

1. LOÈVE, M. Probability Theory, Vol. I, 4th ed., Springer-Verlag, New York, 1977.

2. JAMISON, B.; OREY, S. and PRUITT, W. Convergence of Weighted Averages of Independent Random Variables, Z. Wahrsch. verw. Gebiete 4 (1965), 40–44.

3. ADLER, A. and ROSALSKY, A. On the Strong Law of Large Numbers for Normed Weighted Sums of I.I.D. Random Variables, Stochastic Anal. Appl. 5 (1987), 467-483.

4. ADLER, A. and ROSALSKY, A. On the Chow-Robbins "Fair" Games Problem, Bull. Inst. Math. Academia Sinica 17 (1989), 211-227.

5. KLASS, M. and TEICHER, H. Iterated Logarithm Laws for Asymmetric Random Variables Barely With or Without Finite Mean, Ann. Probab. 5 (1977), 861-874.

6. CHOW, Y.S. and TEICHER, H. Probability Theory: Independence, Interchangeability, Martingales, 2nd ed., Springer-Verlag, New York, 1988.

7. KNOPP, K. Theory and Application of Infinite Series, 2nd English ed., Blackie and Son, London, 1951.

8. SENETA, E. Regularly Varying Functions, Lecture Notes in Math. 508, Springer-Verlag, Berlin, 1976.

9. FELLER, W. An Introduction to Probability Theory and Its Applications, Vol. II, 2nd ed., John Wiley, New York, 1971.

10. ADLER, A.B. Some Limit Theorems for Weighted Sums of Random Variables, Ph.D. dissertation, Dept. of Statistics, Univ. of Florida, Gainesville, Florida, 1987.

11. ADLER, A. Generalized One-Sided Laws of the Iterated Logarithm for Random Variables Barely With or Without Finite Mean, J. Theoretical Probab. 3 (1990).

12. FELLER, W. A Limit Theorem for Random Variables with Infinite Moments, Amer. J. Math. 68 (1946), 257-262.