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RESEARCH NOTES

PROPERTIES OF α-EXPANSIONS OF TOPOLOGIES

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ABSTRACT. The results of O. Njåstad for α-topologies together with the results of P. L. Sharma for anti-compact perfect Hausdorff spaces are combined to produce several counterexamples in one space.

KEY WORDS AND PHRASES. α-topology, nowhere dense set, crowded space, anti-compact space.
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By an enlargement of the usual topology on the unit interval of real numbers, [0,1], several striking counterexamples are obtained at once. O. Njåstad [1] introduced and studied the α-topology τ^{α} for an arbitrary topological space (X,τ). This topology can be defined as the smallest expansion of τ for which the nowhere dense subsets of X relative to τ are closed. Njåstad [1] noted that $\tau^{\alpha} = \{U - N | U \in \tau \text{ and } N \text{ is nowhere dense}\}$, and Andrijević [2] found $\text{Int}^{\alpha}\text{Cl}^{\alpha}A = \text{Int Cl } A$ for every $A \subseteq X$ where $\text{Int}(\text{Int}^{\alpha})$ and $\text{Cl}(\text{Cl}^{\alpha})$ are the interior and closure operators for $\tau(\tau^{\alpha})$. Letting X^{α} be the space X with topology τ^{α}, it follows immediately that X and X^{α} have the same regular open sets, nowhere dense sets, meager sets, clopen sets, and semiopen sets. [3] Recall that a set A is τ-semiopen if $A \subseteq \text{Cl Int } A$. By X_S, the semiregularization of X, is meant the space X with topology τ_S having the τ-regular open sets as a base. Clearly, X and X^{α} have the same semiregularization. Topological properties mutually shared by X and X^{α} are called α-topological and are known [4] to coincide with the semitopological properties of Crossley and Hildebrand [5]. Among these is Baireness. Resolvability and separability are also α-topological since X and X^{α} share the same dense sets (Proposition 1). Of course, the α-topological properties include the semiregular properties shared by a space and its semiregularization. Some important semiregular properties are: Hausdorff separation, Urysohn separation, extremally disconnectedness, H-closedness, pseudocompactness, connectedness, and almost regularity. A space X is almost regular if and only if X_S is regular. Clearly X is regular if and only if X is both semiregular ($X = X_S$) and almost regular.

PROPOSITION 1. The spaces X and X^{α} share the same dense sets.

PROOF: Since $\tau \subseteq \tau^{\alpha}$, each dense subset of X^{α} is dense in X. Suppose that D is a dense subset of X and $W \in \tau^{\alpha}-\{\emptyset\}$. Then $W = U-N$ for some $U \in \tau$ and N, nowhere dense. Since $U\text{-Cl } N \in \tau-\{\emptyset\}$, $(U\text{-Cl } N) \cap D \neq \emptyset$, so that $W \cap D \neq \emptyset$. Evidently D is a dense subset of X^{α}. □

By a theorem of P. L. Sharma [6], if the nowhere dense subsets of a crowded (without isolated points) Hausdorff space are closed then countably compact subsets of the space are finite, i.e. the space is anti-countably compact in the sense of P. Bankston [7] and therefore also anti-compact since compact subsets are finite. Further, such a space is nowhere locally compact [8] since interiors of compact subsets are empty. To use Sharma's result we note the following, whose proof is an easy exercise.

PROPOSITION 2. The space X^{α} is crowded if and only if X is crowded.

By considering two-point Sierpinski space, we see that crowdedness (like Baireness, resolvability, and separability) is an α-topological property which is not semiregular. In the remainder of this article [0,1] is the unit interval of real numbers with the usual subspace topology. We now have the following examples.

EXAMPLE 1. The space $[0,1]^\alpha$ is a crowded, Hausdorff, nonsemiregular, almost regular, nowhere locally compact, Baire space.

EXAMPLE 2. The space $[0,1]^\alpha$ is a separable Hausdorff space which is not a k-space.

EXAMPLE 3. The space $[0,1]^\alpha$ is pseudo-compact but anti-countably compact.

Since the identity function from X^α to X is continuous, connected subsets of X^α are connected in X. But in general, it is not known whether connected subsets of X are connected in X^α. However, the connected subsets of the space [0,1] are the intervals which if not singleton are semi-open. Therefore, by showing that for semi-open subsets A of X, $(\tau|A)^\alpha = \tau^\alpha|A$, from connectedness being α-topological, it will follow that the connected subsets of $[0,1]^\alpha$ are precisely those of [0,1].

PROPOSITION 3. For any subset $A \subseteq X$, $(\tau|A)^\alpha \subseteq \tau^\alpha|A$.

PROOF: If $W \in (\tau|A)^\alpha$, then $W = (U \cap A) - N$ where $U \in \tau$ and N is a nowhere dense subset of $(A, \tau|A)$. Since N is nowhere dense in X we have that $W = (U - N) \cap A \in \tau^\alpha|A$. □

I. L. Reilly and M. K. Vamanamurthy [9] have shown that the reverse of the inclusion of Proposition 3 holds if A is semi-open in X. By the foregoing remarks, the connected subsets of $[0,1]^\alpha$ are the intervals. But since $[0,1]^\alpha$ is anti-compact, the continuous functions from [0,1] into $[0,1]^\alpha$ are constant since each has a nonempty connected and compact image.

EXAMPLE 4. Intervals are connected in the totally path disconnected space $[0,1]^\alpha$.

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