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ON POINT - DISSIPATIVE SYSTEMS OF DIFFERENTIAL EQUATIONS WITH QUADRATIC NONLINEARITY

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ABSTRACT. The system $x' = Ax + f(x)$ of nonlinear vector differential equations, where the nonlinear term $f(x)$ is quadratic with orthogonality property $x^Tf(x) = 0$ for all x, is point-dissipative if $u^TAu < 0$ for all nontrivial zeros u of $f(x)$.

KEY WORDS AND PHRASES. *Point-dissipative, quadratic nonlinearity, symmetric matrices, commutative but generally non-associative algebra.*

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I. INTRODUCTION.

We are concerned with a class of nonlinear vector equations of the form

$$x' = Ax + f(x) \tag{1.1}$$

where the nonlinear term $f(x)$ is quadratic of the form

$$f(x) = \begin{bmatrix} x^TC_1x \\ \cdot \\ \cdot \\ x^TC_nx \end{bmatrix}$$

The $n \times n$ matrices $\{C_i\}$ are symmetric with the orthogonality property

$$x^Tf(x) = 0 \tag{1.2}$$

for all x.

We are interested in investigating the conditions on the $n \times n$ matrix A and $f(x)$ so that the system is point-dissipative, i.e., there is a bounded region which every trajectory of the system eventually enters and remains within.

II. DEFINITIONS.

For each vector $\alpha^T = (\alpha_1, \alpha_2, \ldots \alpha_n)$, we define the matrix $C(\alpha)$ as follows:

$$C(\alpha) = \sum_{i=1}^{n} \alpha_i C_i - \frac{A + A^T}{2} \tag{2.1}$$

The mapping xQy: $R^n \times R^n \to R^n$, where

$$xQy = \begin{pmatrix} x^T C_1 y \\ \vdots \\ x^T C_n y \end{pmatrix} \tag{2.2}$$

can be regarded as a commutative multiplication in R^n. Note that

$$f(x) = xQx$$

$$f(c_1 x) = c_1\, xQc_1 x = c_1^2 xQx = c_1^2 f(x)$$

and the quadratic formula

$$f(c_1 u_1 + c_2 u_2 + c_3 u_3) = \sum_{i,j=1}^{3} c_i c_j u_i Q u_j \tag{2.3}$$

is true for all vectors u_1, u_2, u_3 and all scalars c_1, c_2, c_3.

In addition to the standard vector addition and scalar multiplication in R^n, this multiplication xQy gives the vector space R^n an additional structure of a commutative but generally non-associative algebra B. The algebra B is determined uniquely by the symmetric $n \times n$ matrices $\{C_i\}$. This algebra has been studied by many specially by Markus [1], Gerber, [2], and Frayman [3].

Some algebraic properties of this algebra B will be used to investigate the conditions for point-dissipativeness of the system (1.1). We are specially interested in the concepts of nilpotent and idempotent elements of the algebra B. A nilpotent element $v \neq 0$ satisfies $f(v) = vQv = 0$, while an idempotent element $v \neq 0$ satisfies $f(v) = vQv = v$. It has been proved [3] that in any such algebra B (with or without the orthogonality property $x^T(xQx) = 0$ for all x) generated by any given n symmetric matrices $\{C_i\}$, there exists at least one of these elements.

In our case, because of the orthogonality property (1.2), there cannot exist an idempotent element in the algebra B. For, if $u \neq 0$ is an idempotent, then $0 = u^T f(u) = u^T(uQu) = u^T u = \|u\|^2 \neq 0$ gives us a contradiction. Hence, there must exist at least one nilpotent element in the algebra B. Again by (2.3), a scalar multiple of a nilpotent is also a nilpotent. Hence, the nonlinear quadratic term f(x) in (1.1) has at least one 1-dimensional subspace of zeros.

As an example of system (1.1) with orthogonality property (1.2), we cite the Lorenz system:

$$x' = Ax + f(x) \tag{2.4}$$

where

$$A = \begin{pmatrix} -a & a & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}, \quad a > 0,\ r > 0,\ b > 0$$

$$f(x) = \begin{pmatrix} 0 \\ -xz \\ xy \end{pmatrix}$$

III. LEMMA 1. *If there exists an α so that $C(\alpha)$ is positive definite, then the system $x' = Ax + f(x)$ is point-dissipative.*

The condition on A and f(x) which guarantees the existence of such an α is the topic of our main theorem.

PROOF OF LEMMA 1. Suppose that there exists a vector α such that the matrix $C(\alpha)$ is positive definite. To show that the system (1.1) is point-dissipative, we need to exhibit a bounded region G so that the (positive) trajectory of each solution of (1.1) eventually enters and remains in G. We construct a Lyapunov function of the form

$$V(x) = \frac{1}{2}(x - \alpha)^T (x - \alpha)$$

for which

$$\dot{V}(x) = \alpha^T Ax - x^T C(\alpha)\, x$$

Since the quadratic term $x^T C(\alpha)x$ dominates the linear term $-\alpha^T Ax$, the set

$$S = \{x \mid \dot{V}(x) \geq 0\} \tag{3.1}$$

is bounded. Hence we can choose $r_0 > 0$, sufficiently large, so that the level set (sphere) $V(x) = r_0$ contains in its interior the bounded set S. We choose the interior of the sphere $V(x) = r_0$ to be our bounded region G. Let P_0 be a point outside of G and $\Phi(t, P_0)$ be the solution of (1.1) with $\Phi(0,P_0) = P_0$. Let $V(x) = r_1$ be the level set of V(x) passing through P_0. Clearly $r_1 > r_0$. Let H be the annular closed region formed by the two concentric spheres $V(x) = r_1$ and $V(x) = r_0$. Since the bounded set S lies inside the sphere $V(x) = r_0$, $\dot{V}(x) < 0$ on H. Therefore, $V(\Phi(t, P_0))$ is a decreasing function of t on H. Hence, the trajectory of $\Phi(t, P_0))$ must enter the sphere $V(x) = r_1$ and cannot go outside of the sphere$V(x) = r_1$ at any time $t > 0$. Suppose that the trajectory of $\Phi(t, P_0)$ cannot enter the region G. Then it must remain in H for all time $t \geq 0$. It must have a limit point P in H. By using standard proof we can show that $\dot{V}(P) = 0$ which gives us a contradiction as $\dot{V}(x) < 0$ on H. Hence, the trajectory of $\Phi(t, P_0)$ must eventually enter the bounded region G and cannot go out of G by the decreasing property of $V(\Phi(t, P_0))$ and therefore must remain in G.

IV. THEOREM. *For $n = 2, 3$, the system $x' = Ax + f(x)$ is point-dissipative if and only if $u^T Au < 0$ for all nontrivial zeros u of f(x).*

For $n = 2$, the theorem has already been proved by Bose and Reneke [1]. Hence we will give the proof for $n = 3$. In order to prove the theorem, all we need to show is that the condition $u^T Au < 0$ for all nontrivial zeros of f(x) implies that there exists a vector α such that the matrix $C(\alpha)$ is positive definite. Hence, by Lemma 1, the theorem will be proved.

We also need the following definitions and lemmas:

DEFINITION 1. *Let Z be the set of all zeros of f(x). That is Z contains the zero vector and all the nilpotents of the algebra B.*

DEFINITION 2. *S(u, v) is the 2-dimensional subspace of R^3 generated by two linearly independent vectors u and v.*

DEFINITION 3. *S(u) is the 1-dimensional subspace of R^3 generated by a nontrivial vector u.*

LEMMA 2. *If u is a zero of f(x), then uQx is orthogonal to u for all x.*

LEMMA 3. *If u, v are two linearly independent zeros of f(x), then $S(u, v) \subset Z$ if and only if uQv = 0.*

PROOF OF LEMMA 2. Suppose that u be a zero of f(x). Then by using the quadratic formula (2.3) and the orthogonality relations $(u + x)^T f(u + x) = 0$, $(u - x)^T f(u - x) = 0$, we can show that $u^T(uQx) = 0$, for all x.

PROOF OF LEMMA 3. Let u and v be two linearly independent zeros of f(x). Suppose that uQv = 0. Then $f(c_1u + c_2v) = c_1^2\ uQu + 2c_1c_2\ uQv + c_2^2\ vQv = 0$ implies that $c_1u + c_2v$ is in Z for any two scalars c_1 and c_2. Hence, $S(u, v) \subset Z$. Conversely, suppose that $S(u, v) \subset Z$. Then u + v is in Z and

$$0 = f(u + v) = uQu + 2uQv + vQv = 2\ uQv \text{ implies that } uQv = 0.$$

Let u_1, u_2, u_3 be a basis of R^3, then for any vector $x = d_1 u_1 + d_2 u_2 + d_3 u_3$,

$$x^TC(\alpha)\, x = \alpha^T f(x) - x^TAx = d^T \hat{C}(\alpha)d \tag{4.1}$$

where $d^T = (d_1\ d_2, d_3)$ and the matrix $\hat{C}(\alpha) = ((c_{ij}))$ with

$$c_{ij} = \alpha^T (u_iQu_j) - u_i^TAu_j,\ i, j = 1, 2, 3,$$
$$c_{ij} = c_{ji}$$

Hence, in order to show that the matrix $C(\alpha)$ is positive definite for some α, all we need to show is that the matrix $\hat{C}(\alpha)$ is positive definite for some α.

PROOF OF THE THEOREM. That the condition "$u^TAu < 0$ for all nontrivial u in Z" is necessary follows from (4.1). Hence we need only to show that it is also sufficient.

The proof of the theorem depends on the nature of the set Z of all zeros of f(x). We need to consider the following cases:

Case 1.

(a) Z contains 3 linearly independent vectors with <u>three</u> 2-dimensional subspace of zeros.

(b) Z contains 3 linearly independent vectors with <u>two</u> 2-dimensional subspace of zeros.

(c) Z contains 3 linearly independent vectors with <u>one</u> 2-dimensional subspace of zeros.

(d) Z contains 3 linearly independent vectors with <u>no</u> 2-dimensional subspace of zeros.

Case 2. (a) Z contains 2 linearly independent vectors with one 2-dimensional subspace of zeros.
(b) Z contains 2 linearly independent vectors with no 2-dimensional subspace of zeros.

Case 3. Z contains only one linearly independent vector.

Case 1(a) cannot happen. For suppose that u_1, u_2, u_3 be three linearly independent vector in Z so that $Z = S(u_1, u_2) \cup S(u_1, u_3) \cup S(u_2, u_3)$. Then by lemma 3 $u_iQu_j = 0$, for all i, j = 1, 2, 3. Hence, for any vector $x = c_1u_1 + c_2u_2 + c_3u_3$, $f(x) = \sum_{i,j=1}^{3} c_ic_ju_iQu_j = 0$, implies that $f(x) = 0$, for all x.

Case 1(b) also cannot happen. For suppose that u_1, u_2, u_3 be three linearly independent vectors in Z so that $Z = S(u_1, u_2) \cup S(u_1, u_3) \cup S(u_3)$. Then by lemma 3, $u_iQu_i = 0$, for i = 1, 2, 3, $u_1Qu_2 = 0$, $u_1Qu_3 = 0$ but $u_2Qu_3 \neq 0$. Now $f(u_1 + u_2 + u_3) = 2u_2Qu_3$ and $(u_1 + u_2 + u_3)^T f(u_1 + u_2 + u_3) = 0$ implies that $u_1^T(u_2Qu_3) = 0$. This implies by lemma 2 that u_2Qu_3 is orthogonal to each of the basis vector u_1, u_2, u_3 and hence $u_2Qu_3 = 0$, contradicting our hypothesis.

Case 1(c). Let u_1, u_2, u_3 be three linearly independent vectors in Z so that $Z = S(u_1, u_2) \cup S(u_3)$. Here $u_iQu_i = 0$, i = 1, 2, 3, $u_1Qu_2 = 0$ but $u_1Qu_3 \neq 0$, $u_2Qu_3 \neq 0$. By hypothesis of the theorem

$$(c_1u_1 + c_2u_2)^T A(c_1u_1 + c_2u_2) = \sum_{i,j=1}^{2} c_ic_ju_i^T Au_j$$
$$=(c_1, c_2)\begin{pmatrix} u_1^T Au_1 & u_1^T Au_2 \\ u_1^T Au_2 & u_2^T Au_2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} < 0$$

for all $(c_1, c_2) \neq (0, 0)$. That is

$$\begin{pmatrix} -u_1^T Au_1 & -u_1^T Au_2 \\ -u_1^T Au_2 & -u_2^T Au_2 \end{pmatrix} \text{ is positive definite.}$$

Again u_1Qu_3 and u_2Qu_3 must be linearly independent. For suppose that $c_1(u_1Qu_3) + c_2(u_2Qu_3) = 0$, for some scalars c_1 and c_2. Taking inner product respectively with u_1 and u_2 and using lemma 2, we get

$$c_2u_1^T(u_2Qu_3) = 0$$

$$c_1u_2^T(u_1Qu_3) = 0.$$

Now $u_1^T(u_2Qu_3) = 0$ implies by lemma 2 that u_2Qu_3 is orthogonal to each of the basis vector u_1, u_2, u_3 and hence $u_2Qu_3 = 0$ contradicting our hypothesis that $u_2Qu_3 \neq 0$.

Therefore $u_1^T(u_2Qu_3) \neq 0$, implying that $c_2 = 0$. Similarly $c_1 = 0$. Hence u_1Qu_3 and u_2Qu_3 are linearly independent. We can choose a vector α such that

$$\alpha^T (u_1Qu_3) - u_1^T Au_3 = 0$$

$$\alpha^T (u_2Qu_3) - u_2^T Au_3 = 0.$$

For such a choice of α, the matrix $\hat{C}(\alpha)$ becomes

$$\hat{C}(\alpha) = \begin{pmatrix} -u_1^T Au_1 & -u_1^T Au_2 & 0 \\ -u_1^T Au_2 & -u_2^T Au_2 & 0 \\ 0 & 0 & -u_3^T Au_3 \end{pmatrix}$$

which is positive definite.

Case 1(d). Let u_1, u_2, u_3 be three linearly independent vectors in Z so that $Z = S(u_1) \cup S(u_2) \cup S(u_3)$.
Here $u_iQu_j = 0$, if $i = j$ and $u_iQu_j \neq 0$, if $i \neq j$. As in case 1(c), we can show that u_1Qu_2, u_1Qu_3, u_2Qu_3 are linearly independent. Hence we can choose a vector α such that

$$c_{12} = \alpha^T (u_1Qu_2) - u_1^T Au_2 = 0$$

$$c_{13} = \alpha^T (u_1Qu_3) - u_1^T Au_3 = 0$$

$$c_{23} = \alpha^T (u_2Qu_3) - u_2^T Au_3 = 0.$$

For such a choice of α, the matrix $\hat{C}(\alpha)$ becomes

$$\hat{C}(\alpha) = \begin{pmatrix} -u_1^T Au_1 & 0 & 0 \\ 0 & -u_2^T Au_2 & 0 \\ 0 & 0 & -u_3^T Au_3 \end{pmatrix}$$

which is positive definite.

Case 2(a). Let u_1, u_2 be two linearly independent vectors in Z such that $Z = S(u_1, u_2)$. We can assume that u_1 and u_2 are two unit vectors orthogonal to each other. Let u_3 be a unit vector such that u_1, u_2, u_3 form a orthonormal basis of R^3. Here,

$$u_1Qu_1 = u_1Qu_2 = u_2Qu_2 = 0, \quad u_3Qu_3 \neq 0$$

Since u_1Qu_3 is orthogonal to u_1 and u_3Qu_3 is orthogonal to u_3, we can write

$$u_1Qu_3 = t_1u_2 + t_2u_3, \quad u_3Qu_3 = p_1u_1 + p_2u_2.$$

Using the orthogonality property $(u_1 + u_3)^T f(u_1 + u_3) = 0$, we can show that $p_1 = -2t_2$. Hence, $u_3Qu_3 = -2t_2u_1 + p_2u_2$, $(t_2, p_2) \neq (0, 0)$. Similarly we can show that

$u_2Qu_3 = -t_1u_1 - \frac{1}{2}p_2u_3$. Now, in this case $t_1 = 0$. For $t_1 \neq 0$ implies that

$f\left(-\frac{p_2}{2t_1}u_1 - \frac{t_2}{t_1}u_2 + u_3\right) = 0$. Since $-\frac{p_2}{2t_1}u_1 - \frac{t_2}{t_1}u_2 + u_3$ is not in Z, we get a contradiction. Hence, $u_1Qu_3 = t_2u_3$, $u_2Qu_3 = -\frac{1}{2}p_2u_3$, $u_3Qu_3 = -2t_2u_1 + p_2u_2$. As in case 1(c),

$$\begin{pmatrix} -u_1^T Au_1 & -u_1^T Au_2 \\ -u_1^T Au_2 & -u_2^T Au_2 \end{pmatrix}$$

is positive definite. Taking $\alpha = -\frac{1}{2}\ rt_2\ u_1 + rp_2\ u_2$, where $r > 0$, to be chosen suitably, the matrix $\hat{C}(\alpha)$ becomes

$$\hat{C}(\alpha) = \begin{pmatrix} -u_1^T Au_1 & -u_1^T Au_2 & -u_1^T Au_3 \\ -u_1^T Au_2 & -u_2^T Au_2 & -u_2^T Au_3 \\ -u_1^T Au_3 & -u_2^T Au_3 & r(t_2^2 + p_2^2) - u_3^T Au_3 \end{pmatrix}$$

Here, $\det \hat{C}(\alpha) = r(t_2^2 + p_2^2) \begin{vmatrix} -u_1^T Au_1 & -u_1^T Au_2 \\ -u_1^T Au_2 & -u_2^T Au_2 \end{vmatrix} + \delta$ where δ is a constant (independent of r). Clearly we can choose $r > 0$, sufficiently large, to make $r(t_2^2 + p_2^2) - u_3^T Au_3 > 0$ and $\det \hat{C}(\alpha) > 0$. In other words we can choose a vector α such that $\hat{C}(\alpha)$ is positive definite.

Case 2(b). Let u_1, u_2 be two linearly independent unit vectors in Z so that $Z = S(u_1) \cup S(u_2)$. Let u_3 be a unit vector orthogonal to $S(u_1, u_2)$. Then u_1, u_2, u_3 form a basis of R^3 with $u_3^T u_1 = 0$, $u_3^T u_2 = 0$. Here, $u_1Qu_2 \neq 0$ and $u_3Qu_3 \neq 0$. As in previous cases we can show using lemma 2 and the orthogonality property of f(x) that

$$u_1Qu_2 = s_2u_3,\ s_2 \neq 0,\ u_1Qu_3 = -(t_1u_1^T u_2)u_1 + t_1u_2 + t_2u_3,$$

$$u_3Qu_3 = -(2t_2 + q_2\, u_1^T u_2)u_1 + q_2u_2, \text{ and}$$

$$u_2\, Qu_3 = -\Big(t_1 + \frac{s_2}{1 - (u_1^T u_2)^2}\Big)\, u_1 + \Big(t_1 + \frac{s_2}{1 - (u_1^T u_2)^2}\Big)\, (u_1^T u_2)\, u_2 + \frac{1}{2}\left\{2t_2u_1^Tq_2\left(1 - \left(u_1^T u\right)^2\right)\right\}u_3$$

Now in this case $t_2 = 0$ implies $t_1 = 0$. For, if $t_2 = 0$, then $f\left(\frac{q_2}{2}\ u_1 - t_1\, u_3\right) = 0$ implies that $t_1 = 0$. Hence $t_1 \neq 0$ implies that $t_2 \neq 0$.

In order to prove case 2(b), we also need the following two results:

(i) If $t_1 \neq 0$, then $t_1 + \dfrac{s_2}{1 - (u_1^T u_2)^2} = 0$

(ii) If $t_1 = 0$, then $2t_2\, u_1^T u_2 - q_2\left(1 - \left(u_1^T\ u_2\right)^2\right) \neq 0$

To prove result (i), suppose that $t_1 \neq 0$. We need to show that the vectors u_1Qu_2, u_1Qu_3,

u_2Qu_3 are linearly dependent. Suppose that they are linearly independent.
Then $u_3Qu_3 = c_1(u_1Qu_2) + c_2(u_1Qu_3) + c_3(u_2Qu_3)$ for some $(c_1, c_2, c_3) \neq (0, 0, 0)$. Now

$$f(\tfrac{1}{2}c_2u_1 + \tfrac{1}{2}c_3u_2 - u_3) = (\tfrac{1}{2}c_2c_3 + c_1)(u_1Qu_2) = (\tfrac{1}{2}c_2c_3 + c_1)\, s_2u_3$$

Since $(\frac{1}{2}c_2u_1 + \frac{1}{2}c_3u_2 - u_3)^T f(\frac{1}{2}c_2u_1 + \frac{1}{2}c_3u_2 - u_3) = 0$, we have $\frac{1}{2}c_2c_3 + c_1 = 0$. This in turn implies that $f(\frac{1}{2}c_2u_1 + \frac{1}{2}c_3u_2 - u_3) = 0$ giving us a contradiction. Hence $c_1(u_1Qu_2) + c_2(u_1Qu_3) + c_3(u_2Qu_3) = 0$, for some $(c_1, c_2, c_3) \neq (0, 0, 0)$. That is

$$-\left\{c_2 t_1 u_1^T u_2 + \left(t_1 + \frac{s_2}{1-\left(u_1^T u_2\right)^2}\right)c_3\right\} u_1 + \left\{c_2 t_1 + \left(t_1 + \frac{s_2}{1-\left(u_1^T u_2\right)^2}\right)\left(u_1^T u_2\right)c_3\right\} u_2$$
$$+\left[c_1s_2 + c_2t_2 + \frac{1}{2}c_3\left\{2t_2 u_1^T u_2 - q_2\left(1 - \left(u_1^T u_2\right)^2\right)\right\}\right]u_3 = 0$$

That is $(c_1, c_2, c_3) \neq (0, 0, 0)$ must be a solution of the linear system

$$c_2 t_1\left(u_1^T u_2\right) + \left(t_1 + \frac{s_2}{1-\left(u_1^T u_2\right)^2}\right)c_3 = 0$$

$$c_2 t_1 + \left(t_1 + \frac{s_2}{1-\left(u_1^T u_2\right)^2}\right)\left(u_1^T u_2\right)c_3 = 0$$

$$c_1 s_2 + c_2 t_2 + \frac{1}{2}\left\{2t_2 u_1^T u_2 - q_2\left(1-\left(u_1^T u_2\right)^2\right)\right\}c_3 = 0$$

Now
$$\begin{vmatrix} t_1 u_1^T u_2 & t_1 + \dfrac{s_2}{1-\left(u_1^T u_2\right)^2} \\ t_1 & \left(t_1 + \dfrac{s_2}{1-\left(u_1^T u_2\right)^2}\right)\left(u_1^T u_2\right) \end{vmatrix} = t_1\left(t_1 + \frac{s_2}{1-\left(u_1^T u_2\right)^2}\right)\left(\left(u_1^T u_2\right)^2 - 1\right).$$

Since u_1 and u_2 are two linearly independent unit vectors, $|u_1^T u_2| < 1$ and therefore

$(u_1^T u_2)^2 - 1 \neq 0$. If $t_1 + \dfrac{s_2}{1-\left(u_1^T u_2\right)^2} \neq 0$, then $c_2 = c_3 = 0$. This in turn implies that $c_1 = 0$ contradicting our hypothesis that $(c_1, c_2, c_3) \neq (0, 0, 0)$. Hence $t_1 \neq 0$ implies that

$$t_1 + \frac{s_2}{1-\left(u_1^T u_2\right)^2} = 0.$$

To prove result (ii), suppose that $t_1 = 0$. If $2t_2(u_1^T u_2) - q_2(1 - (u_1^T u_2)^2) = 0$, then

$(2t_2 + q_2(u_1^T u_2))\,(u_1^T u_2) = q_2$, $u_1Qu_2 = s_2u_3$, $s_2 \neq 0$, $u_1Qu_3 = t_2u_3$,

$u_3Qu_3 = \dfrac{-q_2}{u_1^T u_2} u_1 + q_2u_2$ (assuming $u_1^T u_2 \neq 0$) and $u_2Qu_3 = \dfrac{s_2}{1 - \left(u_1^T u_2\right)^2}\left\{- u_1 + \left(u_1^T u_2\right) u_2\right\}$

and $f\left(- \dfrac{q_2}{u_1^T u_2} u_2 + \dfrac{2s_2}{1 - \left(u_1^T u_2\right)^2} u_3 \right) = 0$. Since $s_2 \neq 0$, this implies a contradiction.

Therefore $t_1 = 0$ implies that $2t_2(u_1^T u_2) - q_2(1 - (u_1^T u_2)^2 \neq 0$. In case $u_1^T u_2 = 0$, we can show that $q_2 \neq 0$.

To prove case 2(b), we will consider the following two subcases:

(g) $t_1 \neq 0$, and

(h) $t_1 = 0$.

Consider the subcase (g) first. We have $t_1 \neq 0$, then by result (i) $t_1 + \dfrac{s_2}{1 - \left(u_1^T u_2\right)^2} = 0$.

For this subcase $u_1Qu_2 = s_2u_3$, $s_2 \neq 0$, $u_1Qu_3 = - (t_1u_1^T u_2) u_1 + t_1u_2 + t_2u_3$,

$u_2Qu_3 = \{t_2(u_1^T u_2) - \frac{1}{2} q_2 (1 - (u_1^T u_2)^2)\} u_3$, $u_3Qu_3 = - (2t_2 + q_2 u_1^T u_2) u_1 + q_2u_2$.

Taking $\alpha = k_1u_1 + k_2u_2 + k_3u_3$ the entries c_{ij} of the matrix $\hat{C}(\alpha)$ becomes,

$c_{11} = - u_1^T Au_1$, $c_{22} = - u_2^T Au_2$

$c_{12} = \alpha^T u_1Qu_2 - u_1^T Au_2 = s_2k_3 - u_1^T Au_2$

$c_{13} = \alpha^T u_1Qu_3 - u_1^T Au_3 = t_1(1 - (u_1^T u_2)^2) k_2 + t_2k_3 - u_1^T Au_3$

$c_{23} = \alpha^T u_2Qu_3 - u_2^T Au_3 = \{t_2u_1^T u_2 - \frac{1}{2} (1 - (u_1^T u_2)^2) q_2\} k_3 - u_2^T Au_3$

$c_{33} = \alpha^T u_3Qu_3 - u_3^T Au_3 = -2t_2k_1 - \{2t_2u_1^T u_2 - q_2 (1 - (u_1^T u_2)^2)\} k_2 - u_3^T Au_3$

We can choose k_3 so that $c_{12} = \alpha^T u_1^T Qu_2 - u_1^T Au_2 = 0$. For this k_3 $c_{23} = \alpha^T u_2Qu_3 - u_2^T Au_3 = \text{constant} = \delta$ (say). After choosing k_3, we can now choose k_2 so that $c_{13} = \alpha^T u_1Qu_3 - u_1^T Au_3 = 0$. After choosing k_2 and k_3 in this way, we now choose $k_1 = - \frac{1}{2} t_2r$, where $r > 0$ to be chosen suitably. For such a choice of α, $c_{33} = \alpha^T u_3Qu_3 - u_3^T Au_3 = t_2^2 r + a$ where a is a constant independent of r and the matrix $\hat{C}(\alpha)$ becomes

$$\hat{C}(\alpha) = \begin{pmatrix} - u_1^T Au_1 & 0 & 0 \\ 0 & u_1^T Au_2 & \delta \\ 0 & \delta & t_2^2 r + a \end{pmatrix}$$

Clearly we can choose $r > 0$, sufficiently large to make $\hat{C}(\alpha)$ positive definite. The subcase (h) can be similarly disposed of, using the fact that $t_1 = 0$ implies

$2t_2(u_1^T u_2) - q_2(1 - (u_1^T u_2)^2) \neq 0$.

Case 3. Let u be a unit vector in Z so that Z = S(u). Let u, v, w be an orthonormal basis of R^3. By our assumption vQv ≠ 0 and wQw ≠ 0. Using lemma 2 and the orthogonality property (1.2), we can write

$$uQv = s_1v + s_2w \qquad uQw = t_1v + t_2w$$
$$vQv = -2s_1u + pw \qquad wQw = -2t_2u + qv$$
$$vQw = -(t_1 + s_2)u - \frac{1}{2}pv - \frac{1}{2}qw$$

We will solve this case by considering three subcases:

Subcase (a): $D = \begin{pmatrix} s_1 & t_1 \\ s_2 & t_2 \end{pmatrix}$ is of rank 2

Subcase (b): D is or rank 1

Subcase (c): D is or rank 0

We also need the following two results (i) and (ii):

(i) $t_2 = 0$ implies $t_1 = 0$

(ii) $s_1 = 0$ implies $s_2 = 0$

The result (i) can be proved as in case 2(b). For the result (ii), suppose that $s_1 = 0$ and $s_2 \neq 0$. Then $f(\frac{1}{2}pu - s_2v) = 0$ implies a contradiction. Hence, $s_1 = 0$ implies $s_2 = 0$.

Now consider the subcase (a). The matrix D is non-singular. This implies by (i) and (ii) that $s_1t_2 \neq 0$, otherwise we would get a row of zeros. We will like to show that the quadratic form $x^TDx \neq 0$ for any $x \neq 0$. Suppose that there exists $x^T = (x_1, x_2) \neq (0, 0)$ such that $x^TDx = 0$. Since $s_1t_2 \neq 0$, it follows that $x_1x_2 \neq 0$.

Since D is non-singular, the transpose $D^T = \begin{pmatrix} s_1 & t_1 \\ s_2 & t_2 \end{pmatrix}$ is also non-singular and

$$D^T x = \begin{pmatrix} s_1x_1 + t_1x_2 \\ s_2x_1 + t_2x_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Without loss of generality, suppose that $s_2x_1 + t_2x_2 \neq 0$. Then for any scalar

$$c\, f(cu + x_1v + x_2w) = \{2c(s_2x_1 + t_2x_2) + x_1(x_1p - x_2q)\}\left(-\frac{x_2}{x_1}v + w\right)$$

Since $s_2x_1 + t_2x_2 \neq 0$, we can choose the scalar c, to make $f(cu + x_1v + x_2w) = 0$ contradicting the fact that $x_1x_2 \neq 0$. Hence $x^TDx \neq 0$ for any $x \neq 0$. Therefore by continuity,

$$x^T Dx = x^T \begin{pmatrix} s_1 & \frac{t_1 + s_2}{2} \\ \frac{t_1 + s_2}{2} & t_2 \end{pmatrix} x$$

is either positive definite or negative definite. In either case

$$s_1t_2 - \frac{1}{4}(t_1 + s_2)^2 > 0 \tag{4.2}$$

(4.2) also implies that s_1 and t_2 are of the same sign.

Taking $\alpha = k_1u + k_2v + k_3w$, the entries c_{ij} of the matrix $\hat{C}(\alpha)$ becomes

$$c_{11} = \alpha^TuQu - u^TAu = -u^TAu$$
$$c_{12} = \alpha^TuQv - u^TAv = s_1k_2 + s_2k_3 - u^TAv$$

$$c_{13} = \alpha^T uQw - u^T Aw = t_1k_2 + t_2k_3 - u^T Aw$$
$$c_{22} = \alpha^T vQv - v^T Av = -2s_1k_1 + pk_3 - v^T Av$$
$$c_{33} = \alpha^T wQw - w^T Aw = -2t_2k_1 + qk_2 - w^T Aw$$

$$c_{23} = \alpha^T vQw - v^T Aw = -(t_1 + s_2)k_1 - \frac{1}{2}pk_2 - \frac{1}{2}qk_2 - v^T Aw$$

Since D is non-singular, we can choose k_2 and k_3 so that $c_{12} = c_{13} = 0$. Since s_1 and t_2 are of the same sign, we can choose k_1 with $|k_1|$ sufficiently large to make $c_{22} > 0$, $c_{33} > 0$ and

$$\begin{vmatrix} c_{22} & c_{23} \\ c_{23} & c_{33} \end{vmatrix} = \left\{4s_1t_2 - (t_1 + s_2)^2\right\} k_1^2 + k_1d_1 + d_2 > 0$$

where d_1 and d_2 are constants. Hence for such a choice of k_1, k_2, k_3 the matrix $\hat{C}(\alpha)$ becomes

$$\hat{C}(\alpha) = \begin{pmatrix} -u^T Au & 0 & 0 \\ 0 & c_{22} & c_{23} \\ 0 & c_{23} & c_{33} \end{pmatrix}$$

which is positive definite.

Now consider the subcase (b). Here rank D = 1. Without loss of generality we can assume that $(t_1, t_2) \neq (0, 0)$. This implies that $t_2 \neq 0$, by property (i). Let $(s_1, s_2) = k(t_1, t_2)$. This implies thatk $= t_1/t_2$. For suppose that $k \neq t_1/t_2$. Then for any scalar c, $f(cu + t_2v - t_1w) = \{2c(kt_2 - t_1) + t_2p + t_1q\}\ (t_1v + t_2w)$

Since $kt_2 - t_1 \neq 0$, we can choose the scalar c so that $f(cu + t_2v - t_1w) = 0$ implying that $t_1 = t_2 = 0$ contradicting our assumption. This also implies that $t_2p + t_1q \neq 0$. Hence

$$D = \begin{pmatrix} t_1^2/t_2 & t_1 \\ t_1 & t_2 \end{pmatrix}$$

With this D

$$uQv = \frac{t_1^2}{t_2} v + t_1w$$
$$uQw = t_1v + t_2w$$

$$vQv = \frac{-2t_1^2}{t_2} u + pw$$
$$wQw = -2t_2u + qv$$

$$vQw = -2t_1u - \frac{1}{2}pv - \frac{1}{2}qw$$

Since $vQv \neq 0$, we have $(t_1, p) \neq (0, 0)$. Taking $\alpha = \frac{1}{2}r_2t_2u + r_1qv + r_1pw$, where $r_1 > 0$, $r_2 > 0$ to be chosen suitably, the entries c_{ij} of the matrix $\hat{C}(\alpha)$ becomes

$$c_{11} = -u^T Au,\ c_{12} = \frac{r_1t_1}{t_2}(t_1q + t_2p) - u^T Av,\ c_{13} = r_1(t_1q + t_2p) - u^T Aw$$

$$c_{22} = r_2t_1^2 + r_1p^2 - v^T Av,\ c_{23} = t_1t_2r_2 - r_1pq - v^T Aw$$

$$c_{33} = r_2t_2^2 + r_1q^2 - w^T Aw$$

Now $c_{11} = -u^TAu > 0$, $\begin{vmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{vmatrix} = r_2\,(-u^TAu)t_1^2 + d_1\,(r_1)$, where $d_1\,(r_1)$ is a quadratic in r_1 and $\det \hat{C}(\alpha) = r_2\,[(-u^TAu)\,(t_1q + t_2p)^2r_1 + d_2] + d_3\,(r_1)$, where d_2 is a constant and $d_3(r_1)$ is a cubic polynomial in r_1. Hence, if $t_1 \neq 0$, then we can choose $r_1 > 0$ large enough to make $-(u^TAu)\,(t_1q + t_2p)^2\,r_1 + d_2 > 0$. After choosing such an $r_1 > 0$, we can choose $r_2 > 0$ sufficiently large to make $\begin{vmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{vmatrix} > 0$ and $\det \hat{C}(\alpha) > 0$. In otherwords we can choose α so that $\hat{C}(\alpha)$ is positive definite.

If $t_1 = 0$, then $\begin{vmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{vmatrix} = (-u^TAu)p^2r_1 + d_4$, where d_4 is a constant and $\det \hat{C}(\alpha) = r_2t_2^2[(-u^TAu)p^2r_1 + d_4] + d_5\,(r_1)$, where $d_5(r_1)$ is a quadratic in r_1. As before we can choose $r_1 > 0$ to make

$$(-u^TAu)\,p^2r_1 + d_4 > 0$$

and after choosing such an $r_1 > 0$, we can choose $r_2 > 0$ to make $\det \hat{C}(\alpha) > 0$. In other words we can choose an α so that $\hat{C}(\alpha)$ is positive definite.

Now consider the subcase (c). Here rank D = 0, which implies that $s_1 = s_2 = t_1\; t_2 = 0$. Hence $uQv = 0$, $uQw = 0$, $vQv = pw$, $p \neq 0$, $wQw = qv$, $q \neq 0$, $vQw = -\frac{1}{2}pv - \frac{1}{2}qw$ and $f(qv + pw) = 0$.

Since $pq \neq 0$, this implies a contradiction. Hence, subcase (c) cannot happen.
This completes the proof.

For an example, the Lorenz system (2.4)

$$x' = Ax + f(x), \qquad \text{where}$$

$$A = \begin{pmatrix} -a & a & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix},\; a > 0,\; r > 0,\; b > 0 \text{ and } f(x) = \begin{pmatrix} 0 \\ -xz \\ xy \end{pmatrix}$$

is point dissipative. The vectors $u = (1, 0, 0)$, $v = (0, 1, 0)$, $w = (0, 0, 1)$ are three linearly independent zeros of $f(x)$ and $Z = S(u) \cup S(v, w)$. The condition $u^TAu < 0$ for all $u \in Z$ can easily be verified.

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