Internat. J. Math. & Math. Sci.
VOL. 14 NO. 1 (1991) 55-68

REFLEXIVITY OF CONVEX SUBSETS OF L(H) AND SUBSPACES OF l^p

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(Received August 17, 1989 and in revised form October 6, 1989)

KEY WORDS AND PHRASES. Reflexive, elementary, transitive, convex set, separation, ℓ^p space.

1980 AMS SUBJECT CLASSIFICATION CODES. 47D15, 52A07, 47D99.

1. INTRODUCTION.

The concepts: reflexive, transitive, and elementary originally arose in invariant subspace theory. It is known that every elementary algebra is 3-reflexive [3] ([2] for more generality), but unknown whether all elementary algebras are 2-reflexive.

This paper is an attempt to generalize the nations of elementarity, reflexivity and transitivity, (as they are defined on page 8 of [1]), to arbitrary convex subsets of L(H) and linear subspaces of ℓ^p. This will lead into understanding of reflexivity as external separation by appropriate linear functionals and elementarity as internal separation. This helps provide perspective on the somewhat mysterious relationship between the two concepts.

2. SEPARATION PROPERTY.

We begin by reviewing the relevant terminology and notation from the classical setting in [1]: $L = L(H)$ being the algebra of all bounded operators on a separable Hilbert space H with inner product denoted by $<\cdot,\cdot>$, S being an arbitrary convex subset of L , $T = T(H)$ being the space of trace class operators, and $F_k = F_k(H)$ being the set of rank k or less operators. For n a positive integer or ∞ , $H^{(n)}$ denotes the direct sum of n copies of H . We also write $<a,t>$ for the trace of the product at when $a \in L$ and $t \in T$. For $a \in L(H)$, $a^{(n)}$ will stand for the operator on $H^{(n)}$ which is a direct sum of n copies of a , and if S is a subset of L(H) , then $S^{(n)} \equiv \{a^{(n)} \in L(H^n) | a \in S\}$. We refer to $a^{(n)}$ and $S^{(n)}$ as *ampliations* of a and S respectively. Given x, y ∈ H , the notation $x \otimes y$ will denote the rank one operator: $z \mapsto \langle z,y \rangle x$. Every operator in $F_1(H)$ has this form and $\langle a, x\otimes y \rangle = \langle ax, y \rangle$ for all $a \in L(H)$. [S] will denote the weak* closure of the linear span of S .

DEFINITION 2.1: Let S be a convex subset of L, $x \in L$, $t \in T$. We say t *separates* x from S if the complex number $\langle x,t \rangle$ does not lie in the closure of $\{\langle a,t \rangle | a \in S\}$. A subset A of T *separates* x from S if some $t \in A$ does.

DEFINITION 2.2: Let S be a convex subset of L, $1 \leq k \leq \infty$.

(1) S is *k-reflexive* if F_k separates each $x \in L \backslash S$ from S .

(2) S is *k-transitive* if F_k does not separate any $x \in L$ from S .

(3) S is *k-elementary* if F_k separates each $x \in S$ from each relatively weak* closed convex subset C of S not containing x .

We will write reflexive for 1-reflexive.

When S is a linear subspace of L , $S_\perp = \{t \in T | t(a) \equiv \langle a,t \rangle = 0 \text{ all } a \in S\}$. In Definition 2.3 below, we establish an analogue of $S_\perp$ for arbitrary convex sets.

NOTE: When S is a linear subspace of L(H) , Definition 2.2 of this paper and [1, Definition 2.1] are equivalent. More details can be found in [2]. In fact Definitions 2.1 and 2.2 are implicit in [2]. The main difference between the approach taken in the present paper from that in [2] is the development of an analogue of $S_\perp$ for arbitrary convex sets in L(H) .

DEFINITION 2.3: (1) For a subset S of L ,

$$S_+ \equiv \{(y,\alpha) \in T \times \mathbf{R} | \operatorname{Re} y(x) \geq \alpha \text{ for all } x \in S\}$$

(2) For a subset $M \subseteq T \times \mathbf{R}$,

$$M^+ \equiv \{x \in L | \operatorname{Re} x(y) \geq \alpha \text{ for all } (y,\alpha) \in M\}$$

NOTATION: $S_+ \# F_k \equiv \{(y,\alpha) \in S_+ | y \in F_k\}$.

In the following proposition, we state some properties of S_+ without proof.

PROPOSITION 2.4: Let S be a subset of L and M a subset of $T \times \mathbf{R}$. Then,

1) S_+ is a norm-closed convex set.

2) M^+ is a weak* closed convex set.

3) $(S_+)^+ = \overline{co}^{w^*}(S)$

4) $S = (S_+)^+$ if and only if S is weak* closed and convex.

5) $(M^+)_+ = \overline{co}(M)$ and $S_+ = ((S_+)^+)_+$

6) If S is a linear manifold in L , then $S_+ = S_\perp \times \mathbf{R}^-$.

PROPOSITION 2.5: Let S be a convex subset of $L(H)$. Then

(1) S is k-reflexive if and only if $(S_+ \# F_k)^+ = S$.

(2) S is k-transitive if and only if $S_+ \# F_k = (0) \times \mathbf{R}^-$.

PROOF: (1) ($\Rightarrow$) Always $S \subseteq (S_+ \# F_k)^+$. Suppose $x_0 \notin S$. By hypothesis, there exists $y_0 \in F_k$ such that $y_0(x_0) \notin \overline{y_0(S)}$; (multiplying by some $\lambda \in \mathbf{C}$ with $|\lambda| = 1$ if necessary), we may assume $\inf_{x \in S} \operatorname{Re} y_0(x) > \operatorname{Re} y_0(x_0)$ for all $x \in S$, so there exists $\alpha_0 \in \mathbf{R}$ such that

$$\operatorname{Re} y_0(x) > \alpha_0 > \operatorname{Re} y_0(x_0) \quad \text{for all} \quad x \in S ,$$

i.e., $(y_0, \alpha_0) \in S_+ \# F_k$ and $x_0 \notin (S_+ \# F_k)^+$.

($\Leftarrow$) Suppose $x_0 \in L\backslash S$ i.e., $x_0 \notin (S_+ \# F_k)^+$. Therefore there exists $(y_0,\alpha_0) \in S_+ \# F_k$ such that $\operatorname{Re} y_0(x) \geq \alpha_0$ for all $x \in S$ and $\operatorname{Re} y_0(x_0) < \alpha_0$; hence $\operatorname{Re} y_0(x_0) \notin \overline{\operatorname{Re} y_0(S)}$, so $y_0(x_0) \notin y_0(S)$ i.e., S is k-reflexive.

(2) ($\Rightarrow$) Since $y(L) \subseteq \overline{y(S)}$ for every $y \in F_k$, then for any $0 \neq y \in F_k$, $y(L) = \overline{y(S)} = \mathbf{C}$ and thus $\operatorname{Re} y$ is bounded below on S iff $y = 0$. We conclude $S_+ \# F_k = \{(y,\alpha) \in F_k \times \mathbf{R} | \operatorname{Re} y(x) \geq a \text{ for all } x \in S\} = \{0\} \times \mathbf{R}^-$

($\Leftarrow$) Suppose S is not k-transitive. Then $y(x_0) \notin \overline{y(S)}$ for some $x_0 \in L\backslash S$ and some $y \in F_k$. Then $y \not\equiv 0$ and (multiplying by some $\lambda \in \mathbf{C}$ with $|\lambda| = 1$ if necessary) $\operatorname{Re} y(x) > \alpha > y(x_0)$ for some $\alpha \in \mathbf{R}$ i.e., $S_+ \# F_k \neq 0 \times \mathbf{R}^-$. □

PROPOSITION 2.6: Let S be a convex subset of $L(H)$. Then,

$$(S_+ \# F_1)^+ = \{a \in L \mid a(z) \in \overline{S(z)} \text{ for all } z \in H\} \equiv B .$$

PROOF: ($\supseteq$) Suppose $a_0 \notin (S_+ \# F_1)^+$. Then there exists $(b_0,\alpha_0) \in S_+ \# F_1$, ($b_0 = z_1 \otimes z_2$ for some $z_1, z_2 \in H$), such that $\operatorname{Re}\langle a(z_2), z_1\rangle > \alpha_0$ for all $a \in S$ but $\operatorname{Re}\langle a_0(z_2), z_1\rangle < \alpha_0$. Thus $a_0(z_2) \notin \overline{S(z_2)}$.

($\subseteq$) $\overline{S(z)}$ is a closed convex subset of H for each $z \in H$. Suppose $a_0 \notin B$; then there exists $z_0 \in H$ such that $a_0(z_0) \notin \overline{S(z_0)}$. Then, by [5, Theorem 3.2.9] and the Riesz Representation Theorem, there exists $y_0 \in H$ such that,

$$\inf_{z \in \overline{S(z_0)}} \text{Re}\langle a(z_0), y_0\rangle > \beta_0 > \text{Re}\,\langle a_0(z_0), y_0\rangle$$

for all $a \in S$ and some $\beta_0 \in \mathbf{R}$. Thus, $\text{Re}(y_0 \otimes z_0)(a) > \beta_0 > \text{Re}(y_0 \otimes z_0)(a_0)$ for all $a \in S$. The first inequality implies $(y_0 \otimes z_0, \beta_0) \in S_+ \# F_1$ and the second inequality implies $a_0 \notin (S_+ \# F_1)^+$. □

THEOREM 2.7: Let S be a convex subset of L(H). Then S is reflexive if and only if $az \in \overline{Sz}$ for all $z \in H$ implies $a \in S$.

PROOF: Apply Proposition 2.5 (1) and 2.6. □

EXAMPLE 2.8: The reflexivity of [S] is neither necessary nor sufficient for the reflexivity of S.

PROOF: To see that the condition is not necessary, let $S = \left\{\begin{pmatrix} t & 1-t \\ 0 & t \end{pmatrix} \mid 0 \le t \le 1\right\}$. Then $[S] = \left\{\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \mid \alpha, \beta \in \mathbf{C}\right\}$. Clearly, [S] is not reflexive. To see that S is reflexive, suppose $bz \in \overline{S(z)}$ for all $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in H$ where $b \equiv \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$. Then given $z_1, z_2 \in \mathbf{C}$, there exists $t \in [0,1]$ satisfying: $b_{11}z_1 + b_{12}z_2 = t\, z_1 + (1 - t)z_2$, and $b_{21}z_1 + b_{22}z_2 = tz_2$. Taking $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we see $b_{21} = 0$ and $0 \le b_{11} \le 1$. Taking $z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we see $b_{12} + b_{22} = 1$. Taking $z = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, we see $b_{11} + b_{12} = 1$. Thus $b_{11} = b_{22}$ and $b_{12} = 1 - b_{22}$ and $0 \le b_{ij} \le 1$ where $i, j = 1, 2$. So $b \in S$ which implies S is reflexive.

To see that the condition is not sufficient, take S such that $S \equiv \left\{\begin{pmatrix} \alpha & 0 \\ \beta & \gamma \end{pmatrix} \mid \alpha + \gamma = 1 \text{ and } \alpha, \beta, \gamma \in \mathbf{C}\right\}$. Then $[S] = \left\{\begin{pmatrix} \alpha & 0 \\ \beta & \gamma \end{pmatrix} \mid \alpha, \beta, \gamma \in \mathbf{C}\right\}$ which is reflexive. However S is not reflexive -- to see that, let $b \equiv \begin{pmatrix} 1/3 & 0 \\ 1/3 & 1/3 \end{pmatrix}$. Then $b \notin S$. But $bz \in \overline{Sz}$ for all $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in H$: If $z_1 = 0$, set $\alpha = 2/3$, $\gamma = 1/3$ and β arbitrary, say $\beta = 0$; then $bz = az \in \overline{Sz}$ where $a \equiv \begin{pmatrix} 2/3 & 0 \\ 0 & 1/3 \end{pmatrix} \in S$. If $z_1 \ne 0$, set $\alpha = 1/3$, $\gamma = 2/3$ and $\beta_0 = 1/3 - \frac{z_2}{3z_1}$; then $bz = cz \in \overline{Sz}$ where $c = \begin{pmatrix} 1/3 & 0 \\ \beta_0 & 2/3 \end{pmatrix} \in S$. □

PROPOSITION 2.9: Let S be a reflexive convex subset of L(H). Then S is elementary if and only if every convex subset of S is reflexive.

PROOF: ($\Rightarrow$) Let A be convex subset of S and $a_0 \notin S$.

Case 1: If $a_0 \in S\backslash A$, since S is elementary, there exists $f = z_1 \otimes z_2 \in F_1$ such that $f(a_0) \notin \overline{f(A)}$. Multiplying by scalar $\lambda \in \mathbf{C}$ (if necessary) with $|\lambda| = 1$, we get $\mathrm{Re}\, f(a) > \beta_0 > \mathrm{Re}\, f(a_0)$ for all $a \in A$ and some $\beta_0 \in \mathbf{R}$. The first inequality implies $(f,\beta_0) \in A_+ \# F_1$ and the second inequality implies $a_0 \notin (A_+ \# F_1)^+$.

Case 2: If $a_0 \in L(H)\backslash S$, then $a_0 \notin (S_+ \# F_1)^+$ which implies $a_0 \notin (A_+ \# F_1)^+$.

($\Leftarrow$) Suppose S is not elementary. Then there exists a convex subset A of S and $a_0 \in S\backslash A$ such that $f(a_0) \in \overline{f(A)}$ for all $f \in F_1$; therefore $a_0 \in (A_+ \# F_1)^+$, so $A \subsetneq (A_+ \# F_1)^+$, so by Proposition 2.5, A is not reflexive. □

PROPOSITION 2.10: Let S be a convex subset of $L(H)$.

(1) S is reflexive if and only if for every $a \in L\backslash S$ there exists $x \in H$ such that $a(x) \notin \overline{S(x)}$.

(2) S is elementary if and only if for every relatively weak* closed convex subset A of S and $a \in S\backslash A$ there exists $x \in H$ such that $a(x) \notin \overline{A(x)}$.

PROOF: (1) is a restatement of Theorem 2.7.

(2) ($\Leftarrow$) Apply [5, Theorem 3.2.9] and the Riesz Representation Theorem to find $f \in F_1$ such that $f(a) \notin \overline{f(A)}$.

($\Rightarrow$) Suppose the conclusion fails i.e., $a(x) \in \overline{A(x)}$ for all $x \in H$. Then $\langle a(x),y\rangle \in \overline{\{\langle bx,y\rangle | b \in A\}}$ for all $x, y \in H$. Thus, $(y \otimes x)(a) \in \overline{(y \otimes x)(A)}$ for all $x, y \in H$, i.e., $f(a) \in \overline{f(A)}$ for all $f \in F_1$. □

COROLLARY 2.11: [1, Corollary 2.4] All ampliations of $L(H)$ are reflexive.

PROPOSITION 2.12: [1, Proposition 2.5].

(1) If $t_1 \in T(H)$, then there exists $t_2 \in T(H^{(k)})$ such that $\langle a,t_1\rangle = \langle a^{(k)},t_2\rangle$ for all $a \in L(H)$ and conversely.

(2) If t_1 in (1) belongs to $F_k(H)$, then t_2 can be chosen to belong to $F_1(H^{(k)})$ and conversely.

PROPOSITION 2.13: Let S be a convex subset of $L(H)$. Then $(S_+^{(k)} \# F_1)^+ = \{(S_+ \# F_k)^+\}^{(k)}$.

PROOF: Similar to the proof of [1, Proposition 2.7]. □

COROLLARY 2.14: Let S be a convex subset of $L(H)$.

(1) S is k-reflexive if and only if $S^{(k)}$ is reflexive.

(2) S is k-elementary if and only if $S^{(k)}$ is elementary.

PROOF: (1) Apply Proposition 2.13.

(2) ($\Rightarrow$) Suppose S is k-elementary, A is a relatively weak* closed convex subset of S and $a_0^{(k)} \in S^{(k)}\backslash A^{(k)}$. Then $a_0 \in S\backslash A$, so there exists $s \in F_k(H)$ such that $s(a_0) \notin \overline{s(A)}$. But then by Proposition 2.12 there exists $t \in F_1(H^{(k)})$ such that $s(a) = \langle a,s\rangle = \langle a^{(k)},t\rangle$ for all $a \in L(H)$. Thus $t(a_0^{(k)}) \notin \overline{t(A^{(k)})}$.

($\Leftarrow$) Suppose $S^{(k)}$ is elementary, A is a relatively weak* closed convex subset of S and $a_0 \in S\backslash A$. Then $a_0^{(k)} \in S^{(k)}\backslash A^{(k)}$, so there exists $t \in F_1(H^{(k)})$ such that $t(a_0^{(k)}) \notin \overline{t(A^{(k)})}$; Proposition 2.12 implies that there exists $s \in F_k(H)$ such that $s(a) = \langle a^{(k)},t\rangle = t(a^{(k)})$ for all $a \in L(H)$. Thus $s(a_0) \notin \overline{S(A)}$. □

PROPOSITION 2.15: (Stability Properties). Suppose S is a convex subset of $L(H)$.

(1) If S is elementary, so are all of its convex subsets, while if S is transitive, so are all larger convex subsets of $L(H)$.

(2) The following operations on S do not effect the enjoyment of the properties of Definition 2.2: translation, multiplication on the left or right by an invertible operator, replacement of S by S^* .

(3) Ampliations of reflexive convex subsets are reflexive.

(4) Ampliations of elementary convex subsets are elementary.

PROOF: Similar to the proof of [1, Proposition 2.9]. □

PROPOSITION 2.16. Let S be a convex subset of $L(H)$.

(1) S is transitive and reflexive if and only if $S = L(H)$.

(2) If $k \geq \dim(H)$, then $S^{(k)}$ is elementary and its weak* closure is reflexive.

PROOF: (1) ($\Rightarrow$) Suppose S is transitive and reflexive. Then $S_+ \# F_1 = \{0\} \times \mathbf{R}^-$, so

$$S = (S_+ \# F_1)^+ = (\{0\} \times \mathbf{R}^-)^+ = L(H) .$$

($\Leftarrow$) Suppose $S = L(H)$; then $S_+ = (L(H))_+ = \{0\} \times \mathbf{R}^-$. Thus, $S_+ \# F_1 = \{0\} \times \mathbf{R}^-$, i.e., S is transitive. It follows that

$$(S_+ \# F_1)^+ = ((L(H))_+ \# F_1)^+ = L(H) = S$$

which implies S is reflexive.

(2) Since $k \geq \dim(H)$, then $F_k \times \mathbf{R} = T \times \mathbf{R}$, so S is k-elementary which implies that $S^{(k)}$ is elementary. Also $S_+ \# F_k = (S_+ \# T) = S_+$ which implies $(S_+ \# F_k)^+ = (S_+)^+$. Thus the weak* closure of S is k-reflexive. □

SEPARATING VECTORS AND EXAMPLES:

DEFINITION 2.17: $z_0 \in H$ is called a *separating vector* for $S \subseteq L(H)$ if whenever $a, b \in S$ satisfy $a(z_0) = b(z_0)$ then $a \equiv b$.

THEOREM 2.18: Let H be a finite dimensional Hilbert space and S be a (weak*) closed convex subset of $L(H)$. If S has a separating vector then S is elementary and 2-reflexive.

PROOF: Let A be a (relatively) weak* closed convex subset of S and $a_0 \in S \backslash A$. Suppose z_0 is the separating vector for S . Then, by [5, Theorem 3.2.9] and the Riesz Representation Theorem, there exists $y_0 \in H$ such that $\mathrm{Re}\langle a(z_0), y_0\rangle > \beta_0 > \mathrm{Re}\langle a_0(z_0), y_0\rangle$ for all $a \in A$ and some $\beta_0 \in \mathbf{R}$. Set $f \equiv y_0 \otimes z_0$; then $f \in F_1$ and $f(a_0) \notin \overline{f(S)}$. Thus S is elementary.

To see that S is 2-reflexive, suppose $b^{(2)}(x \otimes y) \in S^{(2)}(x \otimes y)$ for all $x, y \in H$. Then for each $y \in H$ there exists $a_y \in S$ such that $a_y(z_0) = b(z_0)$ and $a_y(y) = b(y)$. Since z_0 is a separating vector for S , a_y is independent of y and $b \equiv a_y \in S$. □

EXAMPLE 2.19: The subset $S \equiv \left\{\begin{pmatrix} 0 & \beta \\ \alpha & \gamma \end{pmatrix} \mid \alpha + \beta + \gamma = 1 \text{ and } \alpha, \beta, \gamma \in \mathbf{C}\right\}$ of $M_{2\times2}(\mathbf{C})$ is elementary but [S] is not.

PROOF: S has the separating vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, so it is elementary. To see that [S] is not elementary; note that

$$[S] = \left\{\begin{pmatrix} 0 & \beta \\ \alpha & \gamma \end{pmatrix} \mid \alpha, \beta, \gamma \in \mathbf{C}\right\}$$

and

$$[S]_\perp = \left\{\begin{pmatrix} \delta & 0 \\ 0 & 0 \end{pmatrix} \mid \delta \in \mathbf{C}\right\} .$$

Then

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \notin [S]_\perp + F_1 , \text{ so } [S]_\perp + F_1 \neq M_{2\times2}(\mathbf{C}) .$$

EXAMPLE 2.20: The real subspace $M_{2\times2}(\mathbf{R})$ of $M_{2\times2}(\mathbf{C})$ is elementary as a convex subset of $M_{2\times2}(\mathbf{C})$ because it has the separating vector $\begin{pmatrix} -i \\ 1 \end{pmatrix}$. However $(M_{2\times2}(\mathbf{R}))_\perp = \{0\}$, so $(M_{2\times2}(\mathbf{R}))_\perp + F_1(\mathbf{C}) \neq M_{2\times2}(\mathbf{C})$, so [1, Definition 2.1] cannot be used as the definition of elementary *real* linear spaces.

PROPOSITION 2.21: Let $M \subset L(H)$ be a real linear subspace. Then M is elementary if and only if for every $t \in T$ there exists an $f \in F_1$ such that $\mathrm{Re}\, f(a) = \mathrm{Re}\, t(a)$ for all $a \in M$.

PROOF: $(\Leftarrow)$ Suppose S is a relatively weak* closed convex subset of M and $b \in M \setminus S$. Then there exists $t \in T$ such that $t(b) \notin \overline{t(S)}$. Multiply by some $\lambda \in \mathbf{C}$ (if necessary) where $|\lambda| = 1$ to get $\mathrm{Re}\, t(b) \notin \mathrm{Re}\, \overline{t(S)}$. Then, by hypothesis, find $\imath \in F_1$ whose real part agrees with $\mathrm{Re}\, t$ on S.

$(\Rightarrow)$ Suppose M is elementary and $t \in T$. Define $S \equiv \{a \in M \mid \mathrm{Re}\, t(a) = 0\}$. If $S = M$ take $f \equiv 0$; otherwise choose $b \in M$ such that $t(b) = 1$. Choose $f \in F_1$ such that $f(b) \notin \overline{f(S)}$. Multiplying f by a complex scalar if necessary, $\overline{f(S)} \subseteq i\,\mathbf{R}$; then $\mathrm{Re}\, f(b) \notin \mathrm{Re}\, \overline{f(S)} = \{0\}$. So, $\mathrm{Re}(f|_S) = \mathrm{Re}(t|_S)$. Set, $g \equiv \frac{f}{\mathrm{Re}\, f(b)}$; then $g \in F_1$ and $\mathrm{Re}\, g(a) = \mathrm{Re}\, t(a)$ for all $a \in M$. □

The following example shows the existence of weak* closed convex subsets of $M_{2\times 2}(\mathbf{C})$ which are elementary whose real linear spans are not elementary.

EXAMPLE 2.22: Let $S \equiv \left\{\begin{pmatrix} 0 & \beta \\ \alpha & \gamma \end{pmatrix} \mid \alpha + \beta + \gamma = 1 \text{ and } \alpha, \beta, \gamma \in \mathbf{C}\right\}$. Then S is elementary because it has the separating vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Set $B \equiv$ real span of $S = \left\{\begin{pmatrix} 0 & \beta \\ \alpha & \gamma \end{pmatrix} \mid \alpha + \beta + \gamma \in \mathbf{R} \text{ and } \alpha, \beta, \gamma \in \mathbf{C}\right\}$. Let $t \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in T$; we show that there is no $f \in F_1$ such that

$$\mathrm{Re}(t(a)) = \mathrm{Re}(f(a)) \text{ for all } a \in B. \quad (^*)$$

Suppose $f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \in F_1$. For $a = \begin{pmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in B$, we have $t(a) = \mathrm{tr}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) = a_{21} + a_{12}$, and $f(a) = f_{12} \cdot a_{21} + f_{21} \cdot a_{12} + f_{22} \cdot a_{22}$. If $(*)$ holds, then $\mathrm{Re}(f_{12} \cdot a_{21} + f_{21} \cdot a_{12} + f_{22} \cdot a_{22} - a_{21} - a_{12}) = 0$ for all $a \in B$. Set $a_{21} = 1$ and $a_{12} = a_{22} = 0$ to conclude $\mathrm{Re}(f_{12}) = 1$. Similarly $\mathrm{Re}(f_{21}) = 1$ and $\mathrm{Re}(f_{22}) = 0$. Set $a_{21} = i$, $a_{12} = -i$ and $a_{22} = 1$ to see $\mathrm{Im}(f_{12} + f_{21}) = 0$. Set $a_{12} = 1 + i$, $a_{21} = 1 - i$ and $a_{22} = -1$ to see $\mathrm{Re}(f_{12} - if_{12} + f_{21} + if_{21} - f_{22} - 2) = 0$; this implies $\mathrm{Re}(-if_{12} + if_{21}) = 0$, so $\mathrm{Im}(f_{12}) = \mathrm{Im}(f_{21})$. Thus $\mathrm{Im}(f_{12}) = \mathrm{Im}(f_{21}) = 0$. Set $a_{12} = 0$, $a_{21} = -i$ and $a_{22} = 1 + i$ to see $\mathrm{Re}(-if_{12} + f_{22} + if_{22} + i) = 0$. Since $\mathrm{Re}(i) = \mathrm{Re}(f_{22}) = 0$, then $\mathrm{Im}(f_{22}) = \mathrm{Im}(f_{12}) = 0$. We conclude $f_{12} = f_{21} = 1$ and $f_{22} = 0$ i.e., $f \notin F_1$. □

Example 2.23 presents a real subspace of $L(H)$ which is not elementary and Example 2.24 shows that half of this subspace can be elementary. Also Example 2.24

shows that a convex subset of $L(H)$ can be elementary without the existence of a single $f \in F_1$ that globally separates the subset from its complement; rather $f \in F_1$ depends on the point to be separated from the subset.

EXAMPLE 2.23: The real subspace B defined by $B \equiv \left\{ \begin{pmatrix} 0 & \beta \\ \alpha & \gamma \end{pmatrix} \mid \alpha, \beta \in \mathbf{R} \text{ and } \gamma \in \mathbf{C} \right\}$ is not elementary because there is no $f \in F_1$ such that,

$$\mathrm{Re}(t(a)) = \mathrm{Re}(f(a)) \text{ for all } a \in B \qquad (*)$$

where $t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. To see that, fix f so that $f \equiv \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \in F_1$. If $(*)$ holds for this f and all $a \equiv \begin{pmatrix} 0 & \beta \\ \alpha & \gamma \end{pmatrix} \in B$, then

$$\beta + \alpha = \mathrm{Re}(f(a)) = \beta \cdot \mathrm{Re}(f_{21}) + \alpha \cdot \mathrm{Re}(f_{12}) + \mathrm{Re}(\gamma) \cdot \mathrm{Re}(f_{22}) - \mathrm{Im}(\gamma) \cdot \mathrm{Im}(f_{22})$$

for all $\alpha, \beta \in \mathbf{R}$ and all $\gamma \in \mathbf{C}$. Thus $\mathrm{Im}(f_{22}) = \mathrm{Re}(f_{22}) = 0$, i.e., $f_{22} = 0$. Set $\beta = \mathrm{Re}(\gamma) = \mathrm{Im}(\gamma) = 0$, to see that $\alpha \cdot \mathrm{Re}(f_{12}) = \alpha$ for all $\alpha \in \mathbf{R}$, so $\mathrm{Re}(f_{12}) = 1$. Set $\alpha = \mathrm{Re}(\gamma) = \mathrm{Im}(\gamma) = 0$, to see that $\beta \cdot \mathrm{Re}(f_{21}) = \beta$ for all $\beta \in \mathbf{R}$, so $\mathrm{Re}(f_{21}) = 1$. In particular, $f_{12} \cdot f_{21} \neq 0$ but $f_{11} \cdot f_{22} = 0$ i.e., $f \notin F_1$. □

EXAMPLE 2.24: Let $S \equiv \left\{ \begin{pmatrix} 0 & \beta \\ \alpha & \gamma \end{pmatrix} \mid \mathrm{Im}(\gamma) \geq 0 \text{ and } \alpha, \beta \in \mathbf{R} \right\}$. We show that S is elementary. Suppose A is a (relatively) closed subset of S and $b \in S \backslash A$. We must separate b from A by a rank one linear functional.

By [5, Theorem 3.2.9], there is a linear functional ϕ on $M_{2\times 2}$ whose real part separates A from b i.e., for some fixed $h \geq 0$ we have $\mathrm{Re}\, \phi(b) > h$ while A is contained in the set $\{a \in S \mid \mathrm{Re}\, \phi(a) \leq h\}$. There is no loss of generality in assuming the latter set is A; we also assume ϕ takes the form $\phi\left(\begin{pmatrix} 0 & \beta \\ \alpha & \gamma \end{pmatrix}\right) = c_1\alpha + c_2\beta + c_3\gamma$ for real c_1, c_2. If $c_1 \cdot c_2 = 0$ or $c_3 \neq 0$, there is a λ which makes $f \equiv \begin{pmatrix} \lambda & c_1 \\ c_2 & c_3 \end{pmatrix}$ belong to F_1. Since $\phi|_S = f|_S$, no further argument is necessary in this case.

Thus, we are reduced to the case when $A = \left\{ \begin{pmatrix} 0 & \beta \\ \alpha & \gamma \end{pmatrix} \mid c_1\alpha + c_2\beta \leq h \right\}$ with $c_1 \cdot c_2 \neq 0$. Left multiplication by the invertible matrix $\begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$ sends S to itself and preserves separation properties, so we may as well assume $c_1 = c_2 = 1$. Write $b = \begin{pmatrix} 0 & \beta_0 \\ \alpha_0 & \gamma_0 \end{pmatrix}$. Translating S into itself by the matrix $\begin{pmatrix} 0 & \alpha_0\text{-}h \\ -\alpha_0 & -\mathrm{Re}\, \gamma_0 \end{pmatrix}$ allows us to assume further that $h = \alpha_0 = \mathrm{Re}\, \gamma_0 = 0$; multiplication by the scalar $1/\beta_0$ leads to our final reduction:

$$A = \left\{ \begin{pmatrix} 0 & \beta \\ \alpha & \gamma \end{pmatrix} \mid \alpha + \beta \leq 0 \right\}, \text{ while } b = \begin{pmatrix} 0 & 1 \\ 0 & \delta i \end{pmatrix} \text{ with } \delta \geq 0 .$$

We will complete the proof by showing that the vector $x \equiv \binom{\delta+1}{i}$ separates b from A . Suppose $a = \begin{pmatrix} 0 & \beta \\ \alpha & \gamma \end{pmatrix}$ belonged to S and $ax = bx$. Then $\beta = 1$ and $\alpha(\delta+1) + \gamma \cdot i + \delta = 0$. But $\alpha + 1 \leq 0$ and $\mathrm{Im}(\gamma) \geq 0$ imply $\mathrm{Re}[\alpha(\delta+1) + \gamma \cdot i + \delta] \leq -1$ so this is impossible. □

NOTE: The choice of the vector x depends on b and consequently the choice of $f \in F_1$ such that $f(b) \notin \overline{f(A)}$ depends on b .

3. COMMUTATIVE ANALOGUE OF THE CLASSICAL CASE:

The main results of this section are the characterizations of all k-elementary subspaces of ℓ^p(Theorem 3.2) and all 2-reflexive subspaces of $\mathbf{C}^n$ (Theorem 3.9).

Throughout this section $T = \ell^q$ and $L = \ell^p$ for the dual of ℓ^q where $1 < p \leq \infty$; (also $T = L = \mathbf{R}^n$ or $\mathbf{C}^n$) · $F_k = \{x \in \ell^q \mid x$ has at most k nonzero entries$\}$, and $\{e_i\}$ will denote the standard "basis" for ℓ^p, ℓ^q, $\mathbf{C}^n$ or $\mathbf{R}^n$. Also, throughout this section S will denote a linear subspace of ℓ^p (or $\mathbf{C}^n$ or $\mathbf{R}^n$) ; in the setting, Definition 2.2 of the present paper coincide with [1, Definition 2.1].

Let S and S′ be subspaces of ℓ^p or $\mathbf{R}^n$ or $\mathbf{C}^n$. We say S and S′ are *equivalent* if there exist a permutation matrix π and a diagonal matrix μ such that $S' = \pi\mu S$. This is clearly an equivalence relation. When the underlying space L is finite-dimensional, there is an alternative way to describe equivalence which will prove useful in the sequel. Suppose $x = (x_1,x_2,\ldots) \in \mathbf{R}^n$ or $\mathbf{C}^n$. Exchanging the positions of two entries of x or multiplying one of its entries by a non-zero constant does not change the rank of x . These two types of operations on x will be called *basic operations.* Suppose $A \equiv \{a_1,a_2,\ldots\}$ is a basis for $S_\perp$ where S is a subspace of L (L as above) and suppose the same basic operations are done to all of the a_i ; then it is obvious that the resulting set $B = \{b_1,b_2,\cdots\}$ is a set of independent vectors; we will call B an *equivalent basis* to A . The space $S' = ([B])^\perp$ will be called an *equivalent space* to S . If A is a canonical basis (Definition 3.6 below) for S , then an equivalent basis B will be called an *equivalent canonical basis.*

The proof of the following proposition is left to the reader.

PROPOSITION 3.1: Suppose S and S′ are two equivalent subspaces of ℓ^p (or $\mathbf{R}^n$ or $\mathbf{C}^n$) . Then

(1) S is k-elementary if and only if S′ is k-elementary.

(2) S is k-reflexive if and only if S′ is k-reflexive.

(3) S is k-transitive if and only if S′ is k-transitive.

In the sequel, we will often find it notationally convenient to replace subspaces of L by equivalent subspaces.

THEOREM 3.2: Let S be a subspace of ℓ^p (or $\mathbf{R}^n$ or $\mathbf{C}^n$) . Then S is k-elementary if and only if $\dim(S) \le k$.

PROOF: ($\Leftarrow$) Write $\{e_j\}_{j=1}^{\infty}$ for the standard "basis" of ℓ^q . Since $\dim(S) = \dim(\ell^q/S_\perp)$, at most k of the cosets $\{e_j + S_\perp \mid j \in \mathbf{N}\}$ can be independent, so there is a subset A_0 of $\{e_j\}$ having cardinality at most k with $[A_0] + S_\perp = \ell^q$.

($\Rightarrow$) S is k-elementary implies $S_\perp + F_k = \ell^q$. Define

$$\mathfrak{B} = \left\{ A \subset \{e_j\}_{j=1}^{\infty} \mid \begin{matrix} A \text{ consists of distinct vectors and} \\ \text{is of cardinality } k \,. \end{matrix} \right\}$$

For each $A \in \mathfrak{B}$, $[A] = \sum_{e_i \in A} \mathbf{C} \cdot e_i$ is a subspace of ℓ^p of dimension k . The collection $\mathfrak{B}$ is countable and $\bigcup_{A \in \mathfrak{B}} (S_\perp + [A]) = \ell^q$. Since $S_\perp$ is closed and each [A] is finite dimensional, then $S_\perp + [A]$ is closed for each $A \in \mathfrak{B}$. Therefore, there exists an $A_0 \in \mathfrak{B}$ such that $S_\perp + [A_0]$ is of second category. This implies $S_\perp + [A_0]$ contains interior in ℓ^q [1, Theorem 10.3], whence $S_\perp + [A_0] = \ell^q$, so

$$\dim(S) = \dim(\ell^q/S_\perp) \le \dim([A_0]) = k \,. \ \square$$

REMARK 3.3: The proof of the above theorem shows that a subspace S of ℓ^p is k-elementary if and only if there exists a subset $A_0 \subset \{e_j\}_{j=1}^{\infty}$ of distinct vectors and of cardinality k such that $S_\perp + [A_0] = \ell^q$.

PROPOSITION 3.4: Let S be a subspace of ℓ^p (or $\mathbf{R}^n$ or $\mathbf{C}^n$) . If S is m-elementary, then S is (m+1)-reflexive.

PROOF: Without loss of generality, assume $e_1 + S_\perp, \ldots, e_m + S_\perp$ is a basis for $\ell^q/S_\perp$. Since S is m-elementary, by [2, Proposition 7.5], S is 3m-reflexive i.e. $S_\perp \cap F_{3m}$ is dense in $S_\perp$; let $y \in S_\perp \cap F_{3m}$. Write y as sum of rank one (or less) vectors: $y = \sum_{j=1}^{3m} y_j$. We can write each $y_j = z_j + w_j$ where $z_j \in [e_1, \ldots, e_m]$ and each $w_j \in S_\perp$. Since $y_j \in F_1$, we in fact have

$$w_j \in S_\perp \cap F_{m+1} \text{ so } y = \left(\sum z_j\right) + \left(\sum w_j\right) \in [S_\perp \cap F_{m+1}] \,. \ \square$$

PROPOSITION 3.5: Let S be a subspace of ℓ^p (or $\mathbf{R}^n$ or $\mathbf{C}^n$) . Then S is reflexive if and only if for every $i \in N$ either $e_i \in S$ or $e_i \in S_\perp$.

PROOF: ($\Rightarrow$) S reflexive implies $S_\perp \cap F_1$ spans $S_\perp$. Therefore, if $e_i \notin S_\perp$ then $e_i \notin [S_\perp \cap F_1]$; hence $[S_\perp \cap F_1]$ consists of vectors with zero i^{th} coordinate i.e. every vector in $S_\perp$ has zero in its i^{th} coordinate, so $e_i \in S$.

($\Leftarrow$) The hypothesis implies, for each i either the i^{th} coordinate of each vector in $S_\perp$ is zero or $e_i \in S_\perp$. Hence $S_\perp \cap F_1$ is spans in $S_\perp$. □

DEFINITION 3.6: Let S be a subspace of $\mathbf{R}^n$ (or $\mathbf{C}^n$). Suppose

$$\dim(S_\perp) = \dim(\mathbf{R}^n/S) = k .$$

A *canonical basis* for $\mathbf{R}^n/S$ is a basis of the form $\{e_{i_1}+ S, \dots , e_{i_k}+ S\}$. The dual basis $\{\delta_{i_1}, \dots , \delta_{i_k}\}$ for $S_\perp \equiv (\mathbf{R}^n/S)*$ is called a *canonical basis* for $S_\perp$.

NOTE: The i_jth coordinate of δ_{i_j} is one, while its i_1st, i_2nd, i_3rd, ... , i_{j-1}st, . . , i_{j+1}st, ... , i_kth coordinates are all zeros.

PROPOSITION 3.7: Let S be a subspace of $\mathbf{R}^n$ (or $\mathbf{C}^n$). Suppose $\{\delta_1, \dots ,\delta_k\}$ is a canonical basis for $S_\perp$. Then S is transitive if and only if each δ_i has rank > 1.

PROOF: ($\Rightarrow$) Clear.

($\Leftarrow$) The hypothesis implies $\operatorname{rank}(\delta_i) \geq 2$ for all $i = 1, \dots , k$. By definition of canonical basis, any non-trivial linear combination of δ_i's generates a vector of rank 2 or more. Hence S is transitive. □

PROPOSITION 3.8: Let S be a subspace of $\mathbf{R}^n$ (or $\mathbf{C}^n$). Suppose $\{\delta_1, \dots , \delta_k\}$ is a canonical basis for $S_\perp$. Then S is reflexive if and only if each δ_i has rank one.

PROOF: ($\Leftarrow$) Clear.

($\Rightarrow$) Suppose for some i_0, δ_{i_0} has rank ≥ 2. By the definition of the canonical basis, each non-trivial linear combination of δ_{i_0} with any one (or more) of the other δ_i's generates a vector of rank ≥ 2. Therefore,

$$\delta_{i_0} \notin [S_\perp \cap F_1] ,$$

i.e., S is not reflexive. □

THEOREM 3.9: Let S be a subspace of $\mathbf{R}^n$ (or $\mathbf{C}^n$). Suppose $\{\delta_1, \dots , \delta_k\}$ is a canonical basis for $S_\perp$. Then S is 2-reflexive if and only if each δ_i has rank ≤ 2.

PROOF: ($\Leftarrow$) Clear.

($\Rightarrow$) Suppose some δ_i , say δ_1 , has rank ≥ 3 . Apply Proposition 3.1, we may assume $\delta_1 = (1,0,\ldots,0,1,1,*,\ldots,*)$ - the one's are in the positions 1, $k+1$, and $k+2$, and asterisks denote arbitrary numbers. We classify δ_i's according to their entries in the positions $k+1$ and $k+2$ as follows:

$A_1 \equiv \{\delta_i \mid \delta_i$ has (at most) one non-zero in $k+1$ position or $k+2$ position or the entries in these positions are equal$\}$

$A_2 \equiv \{\delta_i \mid \delta_i \notin A_1\}$

Note that if x_i is a non-zero vector in $[A_i]$ for $i = 1, 2$, then $x_1 + x_2$ has rank at least three.

It follows that $S_\perp \cap F_2 \subseteq \bigcup_{j=1}^{2} ([A_j] \cap F_2)$. Hence,

$$\dim([S_\perp \cap F_2]) \leq \dim([[A_1] \cap F_2]) + \dim([[A_2] \cap F_2]) \leq \dim(S_\perp) .$$

If $\dim([A_1]) = m_1$ and $\dim([A_2]) = m_2$, then $m_1 + m_2 \leq k$; also note that $\dim([[A_1] \cap F_2]) \leq m_1 - 1$. Therefore,

$$\dim([S_\perp \cap F_2]) \leq m_1 + m_2 - 1 \leq k - 1 < k = \dim(S_\perp) ,$$

i.e., S is not 2-reflexive. □

REMARK: 3.7-3.9 do *not* generalize to easily stated characterizations of k-transitive subspaces of $\mathbb{C}^n$ for $k \geq 2$ or k-reflexive subspaces of $\mathbb{C}^n$ for $k \geq 3$.

(Research partially supported by Southern Arkansas University Research Fund.)

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