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PROPERTY Q

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ABSTRACT. Some properties of property Q are stated, some new results are proved and implications to totally metacompact and totally paracompact are obtained.

KEY WORDS AND PHRASES. Property Q, metacompact, totally metacompact, totally paracompact.

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1. INTRODUCTION.

An open cover has **property Q** [1] if when $\{O_i : i \in N\}$ is a sequence of distinct members of the cover and p_i, q_i are points of O_i and $\{p_i\}$ has limit p, then $\{q_i\}$ has limit p. A topological space has **property Q** if each open cover has an open refinement having property Q. A topological space is **metacompact** if each open cover has a point finite refinement that covers the space. A topological space is **totally paracompact (totally metacompact)** if each open base contains a locally finite (point finite) subcover. A basis is a **uniform base** if each infinite collection from the basis containing a point is a basis at the point. All spaces are assumed to be Hausdorff topological spaces. Some previous results pertaining to property Q are:

THEOREM 1. [1] A complete Moore space that satisfies property Q is a metric space.

THEOREM 2. [2] A space that satisfies property Q is metacompact.

THEOREM 3. [2] A first countable space that satisfies property Q is paracompact.

THEOREM 4. [3] A developable space is metrizable if and only if it satisfies property Q.

It follows that a countably compact space satisfying property Q is compact and that an M-space satisfying property Q is metric.

2. RESULTS.

DEFINITION 1. A basis β is a **Q base** if β satisfies property Q.

LEMMA 1. If the space X has a Q base, then X has a uniform base.

PROOF. If X has the discrete topology, then the lemma is true. Therefore let Y be the set of nondiscrete points of X and B be a Q base for Y. For any infinite subcollection β of B containing a point p, we need to show that β is a basis at p. Suppose not, then there is an open set O containing p that contains no member of β. Select a countably infinite subcollection $\{B_i : i \in N\}$ of β containing p, and choose points $\{p_i\}$ from distinct members of $\{B_i\}$ but not in O. Then $\{p_i\}$ must have sequential limit p because β has property Q. This is a contradiction.

THEOREM 5. A space is metric if and only if it is a regular space with a Q base.

PROOF. Note that a regular space with a uniform base is developable [4]. And a regular developable space satisfying property Q is metric [3].

Conversely a metric space has a Q base. For each integer n, use locally finite refinements of balls with diameters less than $1/n$.

DEFINITION 2. A topological space is **totally** Q if each open base contains a subcover satisfying property Q.

THEOREM 6. If X is totally Q, then X is totally metacompact.

PROOF. Let B be a basis for X. Then there is a subcollection β of B covering X and having property Q. Well order β and let B_1 be the first member in this ordering. And let B_α be the first member of the well ordering that contains a point not in $\cup_{\beta<\alpha}B_\beta$. Claim $\{B_\alpha\}$ is point finite. Suppose that p is point in infinitely many members of $\{B_\alpha\}$; then we pick a countably infinite subsequence of sets $\{B_{\alpha_i}\}$ from $\{B_\alpha\}$ each containing p. Let p_1 be a point in B_{α_1}, then from each B_{α_i} we choose a point p_i not in $\cup_{j<i}B_{\alpha_j}$. Then $\{p_i\}$ has p as sequential limit by property Q but B_{α_1} is an open set containing p but no point of $\{p_i : i > 1\}$ a contradiction.

The converse of Theorem 6 is not true. Let X and Y be one-point compactifications of discrete spaces of size ω and ω_1, then the space $X \times Y - \{(\omega, \omega_1)\}$ with the product topology is totally metacompact but not totally Q.

THEOREM 7. A first countable, totally Q space X is totally paracompact.

PROOF. Let B be a basis for X. By Theorem 6 there is a subcollection β of B that is point finite and minimal (minimal in the sense that if b is in β then b is not a subset of any other member of β).

Claim β is locally finite. Suppose not, then there is a point p of X so that each open set containing p intersects infinitely many members of β. Let B_o be one of the finitely many members of β containing p. Let $\{O_i\}$ be a countable basis at p. Then for each natural number i, choose $B_i \in \beta$ such that $B_i \cap O_i$ in not empty, and the B_i's are distinct members of β which are also different from B_o. For each i, choose p_i in $B_i \cap O_i$ and $q_i \in (B_i - B_o)$. Since $\{p_i\}$ has sequential limit point p; therefore, $\{q_i\}$ must have sequential limit point p by property Q. This is a contradiction; hence, $\{B_\alpha\}$ is a locally finite subcollection of β.

Example 2.14 in [3] is an example of a totally Q space that is not totally paracompact. In [5] it is proved that a locally compact space is paracompact if and only if it is mesocompact. It is not true that a locally compact space is paracompact if and only if it satisfies property Q.

EXAMPLE. A locally compact property Q space that is not paracompact.

Let $\beta\omega$ and $\beta\omega_1$ be the Stone-Čech compactifications of discrete spaces of size ω and ω_1. Then the space

$$\beta\omega \times \beta\omega_1 - (\beta\omega - \omega) \times (\beta\omega_1 - \omega_1),$$

with the topology inherited as a subspace of the product space $\beta\omega \times \beta\omega_1$, has the desired property.

An open cover has **strong property Q** if it has property Q and when $\{p_i\}$ has cluster point p, then $\{q_i\}$ has cluster point p. A topological space is strong property Q if each open cover has a refinement satisfying strong property Q.

THEOREM 8. A regular, locally compact, strong property Q space is paracompact.

PROOF. First note that a regular, locally compact, strong property Q space is metacompact. And suppose we have a regular, locally compact, strong property Q space that is not paracompact. Then there is an open cover O and a point p so that every open refinement of O is not locally finite at p. Let R be a point finite minimal open refinement of O satisfying strong property Q. Let C be an open set containing p so that C is a subset of some member of R and the closure of C is compact. Let G be an open set containing p so that the closure of G is a subset of C. The set G must intersect infinitely many members of R and each member of R that intersects G must have a point in the complement of C (otherwise R would not be minimal). Hence, sequences $\{p_i\}$ and $\{q_i\}$ exist with p_i in G and q_i not in C and $\{p_i\}$ must have a cluster point that can not be a cluster point of $\{q_i\}$. This is a contradiction. Therefore, the space must be paracompact.

QUESTION When does totally Q imply totally paracompact?

REFERENCES

1. YOUNGLOVE, J. N. Concerning dense metric subspaces of certain nonmetric spaces, Fund. Math. 48(1960), 15–25.

2. BRIGGS, R. C. A Comparison of Covering Properties in T3 and T4 Spaces, Ph. D. University of Houston, 1968.

3. MANCUSO, V. J. Mesocompactness and related properties, Pacific J. Math. 33(1970), 345–355.

4. ALEKSANDROV, P. S. On the metrization of topological spaces, Bull. Acad. Polon. Sci. Ser. Sci Math. Astronom. Phys. 8(1960), 135–140.

5. BOONE, J. R. Some characterizations of paracompactness in K–spaces, Fund. Math.72(1971), 145–153.