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A PARSEVAL-GOLDSTEIN TYPE THEOREM ON THE WIDDER POTENTIAL TRANSFORM AND ITS APPLICATIONS

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ABSTRACT. In this paper a Parseval-Goldstein type theorem involving the Widder potential transform and a Laplace type integral transform is given. The theorem is then shown to yield a relationship between the $\mathcal{K}$-transform and the Laplace type integral transform. The theorem yields some simple algorithms for evaluating infinite integrals. Using the theorem and its results, a number of new infinite integrals of elementary and special functions are presented. Some illustrative examples are also given.

KEY WORDS AND PHRASES. The Widder potential transform, the Laplace transform, the $\mathcal{L}_2$-transform, the $\mathcal{K}$-transform the modified Bessel function of the third kind or Macdonald's function.
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1. INTRODUCTION

Widder [1,2] presented a systematic account of the potential transform

$$\mathcal{P}[f(x);y]=\int_0^\infty \frac{xf(x)}{x^2+y^2}\,dx\,. \tag{1.1}$$

Widder pointed out that the potential transform is related to the Poisson integral representation of a function which is harmonic in a half plane and gave several inversion formulae for the transform and applied his results to harmonic functions. Srivastava and Singh [3] gave the following Parseval-Goldstein type formula:

$$\int_0^\infty x\,\mathcal{P}[f(u);x]\,g(x)\,dx=\int_0^\infty x\,f(x)\,\mathcal{P}[g(u);x]\,dx \tag{1.2}$$

for the Widder potential transform. Srivastava and Yürekli [4] gave the following Parseval-Golstein type relation involving the Laplace transform, the Fourier sine transform and the

Widder potential transform:

$$\int_0^\infty \mathcal{L}[f(u),x]\,\mathcal{F}_s[g(u),x]\,dx = \int_0^\infty f(x)\,\mathcal{P}[g(u),x]\,dx \tag{1.3}$$

where $\mathcal{L}$ is the Laplace transform and $\mathcal{F}_s$ is the Fourier sine transform. There are numerous analogous results in the literature on integral transforms. (See, for instance, Goldstein [5], Srivastava [6], Srivastava and Panda [7] and Yürekli [8].)

The objective of this paper is to establish a Parseval-Goldstein type relation between the potential transform and a Laplace type integral transform called $\mathcal{L}_2$-transform where the $\mathcal{L}_2$-transform is defined as

$$\mathcal{L}_2[f(x);y] = \int_0^\infty x\,e^{-x^2y^2} f(x)\,dx\,. \tag{1.4}$$

If we make a change of variable in the integral on the right-hand side of (1.4), one obtains

$$\mathcal{L}_2[f(x);y] = \frac{1}{2}\int_0^\infty e^{-xy^2} f(\sqrt{x})\,dx\,. \tag{1.5}$$

Comparing (1.5) with the definition of Laplace transform we obtain the following relationship between the Laplace transform and the $\mathcal{L}_2$-transform

$$\mathcal{L}_2[f(x);y] = \frac{1}{2}\,\mathcal{L}[f(\sqrt{x});y^2]\,. \tag{1.6}$$

We also obtain identities relating the $\mathcal{K}$-transform

$$\mathcal{K}_\nu[f(x);y] = \int_0^\infty \sqrt{xy}\,K_\nu(xy)\,f(x)\,dx \tag{1.7}$$

to the $\mathcal{L}_2$-transform, where $K_\nu(x)$ is the Bessel function of the third kind (it is also known as the Macdonald function), and the Laplace transform to the $\mathcal{L}_2$-transform. Using these results we show how one can extend tables of Laplace and Hankel transforms. (See Erdélyi et al. [9, 10], Oberhettinger [11], Oberhettinger and Badii [12].) For definitions of special functions that are used in the paper, the reader is referred to Oldham and Spanier [13], and Erdélyi et al. [14].

We note that if we write $\mathcal{M}[f(x);y] = F(y)$ where $\mathcal{M}$ represents any integral transform, we mean the $\mathcal{M}$-transform of $f(x)$ exists and it is $F(y)$. ,

2. A PARSEVAL-GOLDSTEIN THEOREM AND ITS COROLLARIES

LEMMA 2.1. We have

$$\mathcal{L}_2[\mathcal{L}_2[f(x);z];y] = \frac{1}{2}\,\mathcal{P}[f(x);y]\,, \tag{2.1}$$

provided that the integrals involved converge absolutely.

PROOF: Using the definition of the $\mathcal{L}_2$-transform we obtain

$$\mathcal{L}_2[\mathcal{L}_2[f(x);z];y] = \int_0^\infty z e^{-y^2z^2}\left(\int_0^\infty x e^{-x^2z^2} f(x)\,dx\right)dz\,. \tag{2.2}$$

Changing the order of integration on the right side of (2.2), which is permissible by the absolute convergence of integrals, we have from (2.2) that

$$\begin{aligned}\mathcal{L}_2[\mathcal{L}_2[f(x);z];y] &= \int_0^\infty x f(x)\left(\int_0^\infty z e^{-(x^2+y^2)z^2}\,dz\right)dx\\ &= \frac{1}{2}\int_0^\infty \frac{x f(x)}{x^2+y^2}\,dx\,.\end{aligned} \tag{2.3}$$

Now the result follows from (1.1) and (2.3).

THEOREM 2.1. We have

$$\int_0^\infty x\,\mathcal{L}_2[f(y);x]\,\mathcal{L}_2[g(z);x]\,dx = \frac{1}{2}\int_0^\infty y\,f(y)\,\mathcal{P}[g(z);y]\,dy\,. \tag{2.4}$$

provided that the integrals involved converge absolutely.

PROOF: Using the definition of the $\mathcal{L}_2$-transform we obtain

$$\int_0^\infty x\,\mathcal{L}_2[f(y);x]\,\mathcal{L}_2[g(z);x]\,dx = \int_0^\infty x\,\mathcal{L}_2[g(z);x]\left(\int_0^\infty y\,e^{-x^2y^2}\,dy\right)dx\,. \tag{2.5}$$

Changing the order of integration, which is permissible by the hypothesis, and then using the definition of the $\mathcal{L}_2$-transform we find that

$$\int_0^\infty x\,\mathcal{L}_2[f(y);x]\,\mathcal{L}_2[g(z);x]\,dx = \int_0^\infty y\,f(y)\,\mathcal{L}_2[\mathcal{L}_2[g(z);x];y]\,dy\,. \tag{2.6}$$

Now the assertion follows from Lemma 2.1.

REMARK 2.1. We have

$$\int_0^\infty x\,\mathcal{L}_2[g(y);x]\,\mathcal{L}_2[f(z);x]\,dx = \frac{1}{2}\int_0^\infty y\,g(y)\,\mathcal{P}[f(z);y]\,dy\,, \tag{2.7}$$

since the relation (2.4) is symmetrical with respect to f and g. Using the relations (2.4) and (2.7) we obtain the Parseval-Goldstein type formula (1.2). Thus, Theorem 2.1 generalizes relation (1.2).

COROLLARY 2.1. We have

$$\int_0^\infty x\,h(x)\,\mathcal{L}_2[f(z);x]\,dx = \int_0^\infty y\,f(y)\,\mathcal{L}_2[h(x);y]\,dy\,. \tag{2.8}$$

provided that the integrals involved converge absolutely.

PROOF: The identity (2.8) follows immediately after letting $h(x) = \mathcal{L}_2[g(z);x]$ in the relation (2.4).

COROLLARY 2.2. We have

$$\mathcal{P}[\mathcal{L}_2[g(u);x];z] = \mathcal{L}_2[\mathcal{P}[g(u);x];z] \tag{2.9}$$

provided that the integrals involved converge absolutely.

PROOF: We set $f(y) = e^{-z^2y^2}$ in Theorem 2.1. Then

$$\begin{aligned}\mathcal{L}_2[f(y);x] &= \int_0^\infty y e^{-(z^2+x^2)y^2}\,dy \\ &= \frac{1}{2(z^2+x^2)}\,.\end{aligned} \tag{2.10}$$

Now the assertion (2.9) follows from (2.4) and (2.10).

THEOREM 2.2. If $\operatorname{Re}\nu \geq -1$

$$\mathcal{K}_\nu[y^{\nu+\frac{1}{2}}f(y);z] = 2^\nu\,z^{\nu+\frac{1}{2}}\,\mathcal{L}_2\left[x^{2\nu-2}\,\mathcal{L}_2\left[f(y);\frac{1}{2x}\right];z\right]. \tag{2.11}$$

provided that the integrals involved are absolutely convergent.

PROOF: We set $g(u) = u^{\nu} J_{\nu}(zu)$ in Theorem 2.1, where J_{ν} is the Bessel function of the first kind of order ν. Using relation (1.6) and then making use of the Laplace transform table (see [9, formula (30), p. 185]) we have

$$\mathcal{L}_2[u^{\nu}\, J_{\nu}(zu); z] = \frac{1}{2}\,\mathcal{L}\left[u^{\frac{\nu}{2}}\, J_{\nu}\left(z\, u^{\frac{1}{2}}\right); x^2\right] = \frac{1}{2}\left(\frac{z}{2}\right)^{\nu} x^{-2\nu-2} \exp\left(-\frac{z^2}{4x^2}\right). \tag{2.12}$$

Now in order to evaluate the potential transform of the function $g(u)$ we use Lemma 2.1 and obtain

$$\mathcal{P}[g(u); y] = \left(\frac{z}{2}\right)^{\nu} \mathcal{L}_2\left[x^{-2\nu-2} \exp\left(-\frac{z^2}{4x^2}\right); y\right]. \tag{2.13}$$

The $\mathcal{L}_2$-transform on the right-hand side of (2.12) may be evaluated by using the relation (1.6) and then the Laplace transform table (see [9, formula (20), p. 146]). Thus

$$\mathcal{P}[g(u); y] = y^{\nu}\, K_{\nu}(zy). \tag{2.14}$$

Substituting the results (2.12) and (2.14) into (2.4) of Theorem 2.1 gives

$$\int_0^{\infty} y^{\nu+1}\, K_{\nu}(zy)\, f(y)\, dy = \left(\frac{z}{2}\right)^{\nu} \int_0^{\infty} x^{-2\nu-1} \exp\left(-\frac{z^2}{4x^2}\right) \mathcal{L}_2[f(y); x]\, dx. \tag{2.15}$$

Now the assertion follows by making the change of variable $x = t/2$ and then by using the definitions of the $\mathcal{K}$-transform and the $\mathcal{L}_2$-transform.

It is well known that

$$\mathcal{K}_{\frac{1}{2}}(x) = \mathcal{K}_{-\frac{1}{2}}(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x}, \tag{2.16}$$

(see [13, p. 306]). Using (2.11) and (2.16) we obtain the identities in the following corollary:

COROLLARY 2.3. We have

$$\mathcal{L}[y f(y); z] = \frac{2}{\sqrt{\pi}}\, z\, \mathcal{L}_2\left[\frac{1}{x}\, \mathcal{L}_2\left[f(y); \frac{1}{2x}\right]; z\right] \tag{2.17}$$

$$\mathcal{L}[y f(y); z] = \frac{1}{\sqrt{\pi}}\, \mathcal{L}_2\left[\frac{1}{x^3}\, \mathcal{L}_2\left[f(y); \frac{1}{2x}\right]; z\right], \tag{2.18}$$

provided that the integrals involved are absolutely convergent.

3. EXAMPLES

We shall illustrate the above results by several examples. In the following example we present a known result (see [10, formula (1), p. 127]) in order to verify relation (2.11).

EXAMPLE 3.1. We show that

$$\mathcal{K}_{\nu}\left[y^{\rho-1}; z\right] = 2^{\rho-\frac{3}{2}} z^{-\rho}\, \Gamma\left(\frac{\rho}{2} - \frac{\nu}{2} + \frac{1}{4}\right) \Gamma\left(\frac{\rho}{2} + \frac{\nu}{2} + \frac{1}{4}\right) \tag{3.1}$$

provided that $\operatorname{Re}\rho > |\operatorname{Re}\nu| - \frac{1}{2}$.

We set $f(y) = y^{\rho-\nu-\frac{3}{2}}$ in Theorem 3.2. Making use of identity (1.6) we obtain

$$\mathcal{L}_2\left[y^{\rho-\nu-\frac{3}{2}}; \frac{1}{2x}\right] = \frac{1}{2}\,\mathcal{L}\left[y^{\frac{\rho}{2}-\frac{\nu}{2}-\frac{3}{4}}; \frac{1}{4x^2}\right] = 2^{\rho-\nu+\frac{1}{2}}\, \Gamma\left(\frac{\rho}{2} - \frac{\nu}{2} + \frac{1}{4}\right) x^{\rho-\nu-\frac{1}{2}} \tag{3.2}$$

provided that $\mathrm{Re}(\rho - \nu) > \frac{1}{2}$. Substituting (3.2) into (2.11) we find

$$\mathcal{K}_\nu\left[y^{\rho-1}; z\right] = 2^\rho - \frac{1}{2}\Gamma\left(\frac{\rho}{2} - \frac{\nu}{2} + \frac{1}{4}\right) z^{\nu+\frac{1}{2}}\mathcal{L}_2\left[x^{\rho+\nu-\frac{3}{2}}; z\right] \tag{3.3}$$

Now formula (3.1) follows after evaluating the $\mathcal{L}_2$-transform on the right side of equation (3.3).

Integral transforms evaluated in Examples 3.2, 3.4 and 3.5, and in the appendix, to the best of the author's knowledge, are all new.

EXAMPLE 3.2. We show that

$$\mathcal{K}_\nu\left[y^{\nu+\frac{1}{2}}\sin ay^2; z\right] = a^{\frac{3}{2}}(2a)^{-\nu-3}\,\Gamma(\nu+2)\,z^{\nu+\frac{3}{2}}\mathrm{S}_{-\nu-\frac{3}{2},\frac{1}{2}}\left(\frac{z^2}{4a}\right). \tag{3.4}$$

where $\mathrm{Re}\,\nu > -2$ and $\mathrm{S}_{\mu,\nu}$ is the Lommel function.

We set $f(y) = \sin ay^2$ in Theorem 3.2. Making use of (1.6) and then tables of Laplace transforms (see [12, formula 7.1, p.54]), we obtain

$$\begin{aligned}\mathcal{L}_2\left[y^{\nu+\frac{1}{2}}\sin ay^2; \frac{1}{2x}\right] &= \frac{1}{2}\mathcal{L}\left[\sin ay; \frac{1}{4x^2}\right]\\ &= \frac{2\alpha x^4}{x^4+\alpha^2}\end{aligned} \tag{3.5}$$

where $\alpha = 1/4a$. Substituting the function f into (2.11), and using (3.5) and (1.6), we find

$$\mathcal{K}_\nu\left[y^{\nu+\frac{1}{2}}\sin ay^2; z\right] = 2^\nu\,\alpha\,z^{\nu+\frac{1}{2}}\,\mathcal{L}\left[\frac{x^{\nu+1}}{x^2+\alpha^2}; z^2\right]. \tag{3.6}$$

Now (3.4) follows from tables of Laplace transforms (see [12, formula 3.11, p.22]).

Using the technique of Example 3.2, we provide additional results in the appendix.

In the following example we obtain a well known result (Erdélyi [9, formula (30) p. 153]) as a special case of Example 3.2.

EXAMPLE 3.3. We show that

$$\mathcal{L}\left[\sin\left(ay^2\right); z\right] = \sqrt{\frac{\pi}{2a}}\left\{\left(\frac{1}{2} - \mathrm{C}(t)\right)\cos t + \left(\frac{1}{2} - \mathrm{S}(t)\right)\sin t\right\}, \tag{3.7}$$

where $t = z^2/(4a)$, and $\mathrm{C}(t)$ and $\mathrm{S}(t)$ are the Fresnel integrals.

We set $\nu = -1/2$ in (3.4). Using (2.16) and the definition of the $\mathcal{K}$-transform we obtain

$$\mathcal{L}\left[\sin ay^2; z\right] = \frac{z}{4a}\mathrm{S}_{-1,\frac{1}{2}}\left(\frac{z^2}{4a}\right). \tag{3.8}$$

It follows from a formula on the Lommel function (see [12, p. 416]) that

$$\mathrm{S}_{-1,\frac{1}{2}}(t) = \pi\left[\mathrm{J}_{\frac{1}{2}}(t) + \mathrm{J}_{-\frac{1}{2}}(t) - \mathbf{J}_{\frac{1}{2}}(t) - \mathbf{J}_{-\frac{1}{2}}(t)\right], \tag{3.9}$$

where $\mathrm{J}_\nu(t)$ is the Bessel function of order ν and $\mathbf{J}_\nu(t)$ is the Anger-Weber function of order ν. However, we have

$$\mathrm{J}_{\frac{1}{2}}(t) = \sqrt{\frac{2}{\pi t}}\sin t \qquad \text{and} \qquad \mathrm{J}_{-\frac{1}{2}}(t) = \sqrt{\frac{2}{\pi t}}\cos t, \tag{3.10}$$

see [13, p. 306] and

$$\mathrm{J}_{\frac{1}{2}}(t) = \sqrt{\frac{2}{\pi t}}\,\{[\mathrm{C}(t) - \mathrm{S}(t)]\cos t + [\mathrm{C}(t) + \mathrm{S}(t)]\sin t\} \tag{3.11}$$

$$\mathrm{J}_{-\frac{1}{2}}(t) = \sqrt{\frac{2}{\pi t}}\,\{[\mathrm{C}(t) + \mathrm{S}(t)]\cos t - [\mathrm{C}(t) - \mathrm{S}(t)]\sin t\}\ , \tag{3.12}$$

see Oberhettinger and Badii [12, p. 415].

Now, substituting (3.10), (3.11) and (3.12) into (3.9) and then using (3.8) we obtain formula (3.7).

EXAMPLE 3.4. We show that

$$\mathcal{L}\left[\left(x + 4ax^3\right)^{\frac{\nu}{2}-\frac{1}{2}}; z\right] = 2^{\nu}(\pi a)^{-\frac{1}{2}}\,\Gamma\left(\frac{\nu}{2} + \frac{1}{2}\right) z^{12} \exp\left(\frac{z}{8a}\right) \mathrm{K}_{\frac{\nu}{2}}\left(\frac{z}{8a}\right)\ , \tag{3.13}$$

provided that $-1 < \operatorname{Re}\nu < 1$, $\operatorname{Re}a > 0$.

We set $f(y) = y^{\nu-1}\exp(-ay^2)$ in Theorem 2.2. Making use of (1.6) and then using tables of Laplace transforms (see [12, formula 5.3, p.37]), we obtain

$$\begin{aligned}\mathcal{L}_2\left[y^{-\nu-1}e^{-ay^2}; \frac{1}{2x}\right] &= \frac{1}{2}\,\mathcal{L}\left[y^{\frac{\nu}{2}-\frac{1}{2}}e^{-ay}; \frac{1}{4x^2}\right]\\ &= 2^{-\nu}\Gamma\left(\frac{1}{2} - \frac{\nu}{2}\right) x^{-\nu}\left(4ax^2 + 1\right)^{\frac{\nu}{2}-\frac{1}{2}}\ ,\end{aligned} \tag{3.14}$$

provided that $\operatorname{Re}\nu < 1$. Using tables of Hankel transforms (see [10, formula (24), p. 132]) we obtain

$$\mathcal{K}_{\nu}\left[y^{-\frac{1}{2}}e^{-ay^2}; z\right] = \frac{1}{4}\sec\left(\frac{\nu\pi}{2}\right)\left(\frac{\pi z}{a}\right)^{\frac{1}{2}} \mathrm{K}_{\frac{\nu}{2}}\left(\frac{z^2}{8a}\right) \tag{3.15}$$

provided that $\operatorname{Re}a > 0$ and $-1 < \operatorname{Re}\nu < 1$. Now formula (3.13) follows from substituting (3.14) and (3.15) into (2.10) and then using (1.6).

EXAMPLE 3.5. We show that

$$\mathcal{L}\left[x^{\frac{1}{2}}\ \mathrm{Erf}\left(ax^{\frac{1}{2}}\right); z\right] = \frac{2}{\sqrt{\pi}}\,z^{-\frac{1}{2}}\arctan\left(az^{-\frac{1}{2}}\right) \tag{3.16}$$

$$\mathcal{L}\left[x^{\frac{3}{2}}\ \mathrm{Erf}\left(ax^{\frac{1}{2}}\right); z\right] = \frac{4}{\sqrt{\pi}}\arctan\left(az^{-\frac{1}{2}}\right)\ . \tag{3.17}$$

where $\mathrm{Erf}(x)$ is the error function.

We set $f(y) = y^{-2}\sin ay$ in Corollary 2.3. Making use of (1.6) and then tables of integral transforms (see [12, formula 7.76, p. 66]) we obtain

$$\begin{aligned}\mathcal{L}_2\left[\frac{1}{y^2}\sin ay; \frac{1}{2x}\right] &= \frac{1}{2}\,\mathcal{L}\left[\frac{1}{y}\sin ay^{\frac{1}{2}}; \frac{1}{4x^2}\right]\\ &= \frac{\pi}{2}\,\mathrm{Erf}(ax)\,.\end{aligned} \tag{3.18}$$

It follows from tables of Laplace transforms (see [12, formula 7.5, p. 54])

$$\mathcal{L}\left[\frac{1}{y}\sin ay; z\right] = \arctan\left(\frac{a}{z}\right)\ . \tag{3.19}$$

Now formula (3.16) follows from substituting (3.18) and (3.19) into (2.17). Similarly, formula (3.17) follows from substituting (3.18) and (3.19) into (2.18) of Corollary 2.3.

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APPENDIX

A. SOME $\mathcal{K}$-TRANSFORM PAIRS

The following formulae (A.1) through (A.5) are consequences of Theorem 2.2. The techniques of Example 3.1 and 3.2 are used to obtain these results.

$$\mathcal{K}_\nu\left[y^{\nu+\frac{1}{2}}\sin ay^2; z\right] = 2(2a)^{-\nu+\frac{1}{2}}\,\Gamma(\nu+2)\,z^{\nu+\frac{3}{2}}\,\mathrm{S}_{-\nu-\frac{3}{2},\frac{1}{2}}\left(\frac{z^2}{4a}\right), \tag{A.1}$$

where $\operatorname{Re}\nu > -2$.

$$\mathcal{K}_\nu\left[y^{\nu+\frac{1}{2}}\cos ay^2; z\right] = 2^{-\frac{1}{2}}(2a)^{-\nu-\frac{1}{2}}\,\Gamma(\nu+1)\,z^{\nu+\frac{3}{2}}\,\mathrm{S}_{-\nu-\frac{1}{2},\frac{1}{2}}\left(\frac{z^2}{4a}\right), \tag{A.2}$$

where $\operatorname{Re}\nu > -1$.

$$\mathcal{K}_1\left[y^{-\frac{1}{2}}\sin ay^2; z\right] = \frac{1}{2}\,z^{-\frac{1}{2}}\left[\mathrm{Ci}\left(\frac{z^2}{4a}\right)\sin\left(\frac{z^2}{4a}\right) - \mathrm{si}\left(\frac{z^2}{4a}\right)\cos\left(\frac{z^2}{4a}\right)\right] \tag{A.3}$$

where Ci(x) and si(x) are cosine and sine integrals, respectively.

$$\mathcal{K}_2\left[y^{-\frac{1}{2}}\sin ay^2; z\right] = z^{-\frac{3}{2}}\left[-\mathrm{Ci}\left(\frac{z^2}{4a}\right)\sin\left(\frac{z^2}{4a}\right) - \mathrm{si}\left(\frac{z^2}{4a}\right)\cos\left(\frac{z^2}{4a}\right)\right] + \frac{1}{4a}z^{-\frac{1}{2}}\left[-\mathrm{Ci}\left(\frac{z^2}{4a}\right)\cos\left(\frac{z^2}{4a}\right) - \mathrm{si}\left(\frac{z^2}{4a}\right)\sin\left(\frac{z^2}{4a}\right)\right]. \quad (A.4)$$

$$\mathcal{K}_\nu\left[y^{\nu+\frac{1}{2}}\sin^2 ay^2; z\right] = 2^{\nu-4}(2a)^{-\nu-\frac{3}{2}}\,\Gamma(\nu+3)\, z^{\nu+\frac{3}{2}}\, \mathrm{S}_{-\nu-\frac{5}{2},\frac{1}{2}}\left(\frac{z^2}{8a}\right), \quad (A.5)$$

where $\mathrm{Re}\,\nu > -3$.

B. SOME LAPLACE TRANSFORM PAIRS

The following formulae (B.1) through (B.5) result from Theorem 2.2 and Corollary 2.3. The techniques of Example 3.4 and 3.5 are used to obtain these results.

$$\mathcal{L}\left[x^{-2\mu}(4ax^2 - x)^{\frac{\nu-1}{2}+\mu}; z\right] = \frac{(4a)^\mu}{z^{\frac{\nu+1}{2}}}\Gamma\left(\frac{\nu+1}{2}-\mu\right)\exp\left(\frac{z}{8a}\right)\mathrm{W}_{\mu,\frac{\nu}{2}}\left(\frac{z}{4a}\right) \quad (B.1)$$

where $2\mathrm{Re}\,\mu < 1 - |\mathrm{Re}\,\nu|$, $\mathrm{Re}\,a > 0$ and $\mathrm{W}_{\mu,\nu}$ is the Whittaker function.

$$\mathcal{L}\left[x^{\frac{\nu}{2}+\frac{\mu}{2}-\frac{3}{4}}\exp\left(\frac{a^2x}{2}\right)\mathrm{D}_{\nu-\mu-\frac{1}{2}}\left(a\sqrt{2x}\right); z\right] = \frac{\pi^{\frac{1}{2}}\,2^{\frac{\mu-\nu}{2}-\frac{5}{4}}\,\Gamma\left(\mu+\nu+\frac{1}{2}\right)}{\Gamma(\mu+1)\left(a+z^{\frac{1}{2}}\right)^{\mu+\nu+\frac{1}{2}}}\;{}_2\mathrm{F}_1\left(\mu+\nu+\frac{1}{2}:,\nu+\frac{1}{2},\mu+1,\frac{a-z^{\frac{1}{2}}}{a+z^{\frac{1}{2}}}\right) \quad (B.2)$$

where $\mathrm{Re}\,\mu > |\mathrm{Re}\,\nu| - \frac{1}{2}$, D_ν is the parabolic cylinder function and ${}_m\mathrm{F}_n$ is the hypergeometric function.

$$\mathcal{L}\left[\frac{\sin\left(\mu\arctan\left(\frac{x}{\alpha}\right)\right)}{x^{\frac{1}{2}}\left(x^2+\alpha^2x\right)^{\frac{\mu}{2}}}; z\right] = \frac{\pi^{\frac{1}{2}}\Gamma(1-\mu)}{4z^{\frac{\mu}{2}+\frac{1}{2}}}\times \left\{\mathrm{J}_{\frac{1}{2}-\frac{\mu}{2}}\left(\frac{\alpha z}{2}\right)\sin\left(\frac{\alpha z}{2}-\frac{\pi}{2}+\frac{\pi}{4}\mu\right) - \mathrm{Y}_{\frac{1}{2}-\frac{\mu}{2}}\left(\frac{\alpha z}{2}\right)\cos\left(\frac{\alpha z}{2}-\frac{\pi}{2}+\frac{\pi}{4}\mu\right)\right\}, \quad (B.3)$$

where $-1 < \mathrm{Re}\,\mu < 2$ and Y_ν is the Bessel function of the second kind of order ν.

$$\mathcal{L}\left[x^{-\frac{1}{2}}\exp\left(\frac{a^2}{4x}\right)\Gamma\left(-\nu,\frac{a^2}{4x}\right); z\right] = \pi\, 2^{2\nu+2}\, a^{\frac{1}{2}}\, z^{-\frac{1}{4}}\,\Gamma\left(\nu+\frac{3}{2}\right)\mathrm{S}_{-2\nu-\frac{3}{2},\frac{1}{2}}\left(az^{\frac{1}{2}}\right), \quad (B.4)$$

where $\mathrm{Re}\,\nu > -1$.

$$\mathcal{L}\left[x^{-\frac{3}{2}}\exp\left(\frac{a^2}{4x}\right)\Gamma\left(-\nu,\frac{a^2}{4x}\right); z\right] = \pi^{\frac{3}{2}}\, 2^{2\nu+3}\, a^{\frac{1}{2}}\,\Gamma\left(\nu+\frac{3}{2}\right)\mathrm{S}_{-2\nu-\frac{3}{2},\frac{1}{2}}\left(az^{\frac{1}{2}}\right), \quad (B.5)$$

where $\mathrm{Re}\,\nu > -1$.