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BOUNDS FOR DISTRIBUTION FUNCTIONS OF SUMS OF SQUARES AND RADIAL ERRORS

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ABSTRACT. Bounds are found for the distribution function of the sum of squares X^2+Y^2 where X and Y are arbitrary continuous random variables. The techniques employed, which utilize copulas and their properties, show that the bounds are pointwise best-possible when X and Y are symmetric about 0 and yield expressions which can be evaluated explicitly when X and Y have a common distribution function which is concave on $(0,\infty)$. Similar results are obtained for the radial error $(X^2+Y^2)^{1/2}$. The important case where X and Y are normally distributed is discussed, and here best-possible bounds on the circular probable error are also obtained.

KEY WORDS AND PHRASES. Bivariate distribution, circular probable error, copulas, radial error, symmetric random variables.
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1. INTRODUCTION.

In a previous paper [1], M. J. Frank and the authors used copulas and their properties to obtain best-possible bounds for the distribution function of the sum X+Y of two random variables X and Y whose individual distribution functions F_X and F_Y are fixed. Our aim in this paper is to apply the techniques developed in [1] to obtain best-possible bounds for the distribution function of the sum of squares X^2+Y^2 and for the so-called radial error $(X^2+Y^2)^{1/2}$ (see Kotz and Johnson [2], p. 499) when the random variables X and Y are continuous. Specifically, we evaluate $\inf[\Pr\{X^2+Y^2 < x\}]$ and $\sup[\Pr\{X^2+Y^2 < x\}]$, where the infimum and supremum are taken over all possible bivariate distribution functions of X and Y having marginals F_X and F_Y. We further show that these bounds are pointwise best-possible when F_X and F_Y are symmetric about 0, and that they are simple to evaluate under the further assumption that X and Y have a common distribution function which is concave on $(0,\infty)$. We also obtain similar results for the radial error and consider the important special case when the margins are normal.

2. PRELIMINARIES.

A (two-dimensional) copula is a mapping C from the unit square $[0,1]^2$ onto the unit interval $[0,1]$ satisfying the conditions:

(i) $C(a,0) = C(0,a) = 0$ and $C(a,1) = C(1,a) = a$ for all a in $[0,1]$.

(ii) $C(c,d) - C(a,d) - C(c,b) + C(a,b) \geq 0$ for all a,b,c,d in $[0,1]$ such that $a \leq c$, $b \leq d$.

Alternatively, a copula is a bivariate distribution function whose support is contained in the unit square and whose margins are uniform on $[0,1]$.

We shall assume that the reader is familiar with the basic facts concerning copulas as given, e.g., in Schweizer and Sklar [3] and reviewed briefly in [1]. Thus, if X and Y are random variables (r.v.'s) with individual distribution functions (d.f.'s) F_X and F_Y and joint d.f. $H_{X,Y}$, then there exists a copula $C_{X,Y}$ such that

$$H_{X,Y}(u,v) = C_{X,Y}(F_X(u),F_Y(v)), \tag{2.1}$$

for all u,v in $\mathbf{R}$. Using (2.1), it follows that the d.f. F_{X+Y} of the sum X+Y may be expressed in the form $\sigma_{C_{X,Y}}(F_X,F_Y)$, where for any copula C, σ_C is the binary operation on the space of (left-continuous) distribution functions defined by

$$\sigma_C(F,G)(x) = \iint_{u+v<x} dC(F(u),G(v)),$$

for any x in $\mathbf{R}$. Also, for any copula C, τ_C and ρ_C are the binary operations defined by

$$\tau_C(F,G)(x) = \sup_{u+v=x} C(F(u),G(v)) \tag{2.2}$$

and

$$\rho_C(F,G)(x) = \inf_{u+v=x} \bar{C}(F(u),G(v)), \tag{2.3}$$

respectively, where $\bar{C}(a,b) = a + b - C(a,b)$ is the dual copula of C.

Let W be the copula defined by

$$W(a,b) = \max(a + b - 1,0), \tag{2.4}$$

so that $\overline{W}(a,b) = \min(a + b,1)$. Then, as shown by Moynihan, Schweizer and Sklar [4],

$$\tau_W(F,G) \leq \sigma_C(F,G) \leq \rho_W(F,G), \tag{2.5}$$

for any d.f.'s F,G and any copula C. In particular,

$$\tau_W(F_X,F_Y) \leq F_{X+Y} \leq \rho_W(F_X,F_Y), \tag{2.6}$$

for any r.v.'s X and Y. Furthermore, as shown in [1], the bounds in (2.5) are pointwise best-possible in the sense that, for any d.f.'s F,G and any x in $\mathbf{R}$,

(i) There exists a copula C_t, dependent only on the value t of $\tau_W(F,G)$ at x, such that

$$\sigma_{C_t}(F,G)(x) = \tau_W(F,G)(x) = t, \tag{2.7}$$

(ii) There exists a copula C_r, dependent only on the value r of $\rho_W(F,G)(x+)$, such that

$$\sigma_{C_r}(F,G)(x+) = \rho_W(F,G)(x+) = r. \tag{2.8}$$

Particular copulas C_t and C_r can be determined explicitly. For example, we may take

$$C_t(a,b) = \begin{cases} \max(a+b-1,t) & (a,b)\in[t,1]^2, \\ \min(a,b), & \text{otherwise}, \end{cases} \tag{2.9}$$

and

$$C_r(a,b) = \begin{cases} \max(a+b-r,0), & (a,b)\in[0,r]^2, \\ \min(a,b), & \text{otherwise}. \end{cases} \tag{2.10}$$

Each of the copulas C_t and C_r assigns all its probability mass to a set of measure zero. Specifically, C_t distributes mass t uniformly on the line segment joining (0,0) to (t,t) and mass $1-t$ uniformly on the line segment joining (t,1) to (1,t); and correspondingly for C_r. As a result, if C_t (or C_r) is the copula of X and Y, then the bivariate distribution $H_{X,Y}$ is singular.

We shall also need the following simple lemmas.

LEMMA 1. If the d.f. F is concave on $(0,\infty)$ and if $F(0) = 0$, then for any x in $\mathbf{R}$,

$$\tau_W(F,F)(x) = \max\{2F(x/2) - 1, 0\}$$

and

$$\rho_W(F,F)(x) = F(x).$$

PROOF. It is readily verified that F is subadditive (see Lemma 2.2.6 of [3]). Thus, for any $u,v \geq 0$, we have

$$F(u+v) \leq F(u) + F(v) \leq 2F((u+v)/2),$$

with equality on the right when $u = v$ and equality on the left when $u = 0$ or $v = 0$. Consequently, for any x in $\mathbf{R}$,

$$\sup_{u+v=x}\{F(u) + F(v)\} = 2F(x/2)$$

and

$$\inf_{u+v=x}\{F(u) + F(v)\} = F(x),$$

from which, using (2.2), (2.3) and (2.4), the lemma follows.

Since the composition of two non-decreasing concave functions is concave, Lemma 1 immediately yields

LEMMA 2. If the d.f. G is concave on $(0,\infty)$ and symmetric about 0, so that $G(-x) = 1 - G(x)$, and if F is defined by

$$F(x) = \begin{cases} 2G(\sqrt{x}) - 1, & x \geq 0, \\ 0, & x \leq 0, \end{cases}$$

then F is a d.f. which is concave on $(0,\infty)$, and

$$\tau_W(F,F)(x) = \begin{cases} \max(4G(\sqrt{x/2}) - 3, 0), & x \geq 0, \\ 0, & x \leq 0, \end{cases}$$

and

$$\rho_W(F,F)(x) = \begin{cases} 2G(\sqrt{x}) - 1, & x \geq 0, \\ 0, & x \leq 0. \end{cases}$$

Note that the d.f. G in Lemma 2 is unimodal and that 0 is its mode (see Feller [5]).

In the remainder of this paper we shall assume that all d.f.'s are continuous (recall that any concave function is continuous). We further note that we use the term "random variable" in the statistical and not the probabilistic sense: that is to say, a r.v. is a quantity whose values are described by a known or unknown probability distribution function rather than a measurable function on a given probability space. (For a detailed discussion of this point of view, see Menger [6].) Also, for any r.v. X, we shall denote the d.f. of X either, as before, by F_X or by df(X), whichever is more convenient.

3. THE BOUNDS.

We now turn our attention to d.f.'s for sums of squares of random variables, that is, to evaluating $\inf[\Pr\{X^2+Y^2 < x\}]$ and $\sup[\Pr\{X^2+Y^2 < x\}]$, where the infimum and supremum are taken over all possible bivariate distribution functions of X and Y having marginals F_X and F_Y. We first note that if Z is a r.v. with d.f. F_Z, then the d.f. of the r.v. Z^2 is given by

$$F_{Z^2}(x) = \begin{cases} F_Z(\sqrt{x}) - F_Z(-\sqrt{x}), & x \geq 0, \\ 0, & x \leq 0. \end{cases} \tag{3.1}$$

Now let X and Y be r.v.'s whose individual d.f.'s F_X and F_Y are specified but whose joint d.f., or equivalently whose copula, is not specified. Then as an immediate consequence of (2.6), we have

THEOREM 1. Let X and Y be continuous r.v.'s. Then for any x in $\mathbf{R}$,

$$\tau_W(F_{X^2},F_{Y^2})(x) \leq df(X^2+Y^2)(x) \leq \rho_W(F_{X^2},F_{Y^2})(x). \tag{3.2}$$

From (2.7) and (2.8) it follows that, for any x in $\mathbf{R}$, there exist copulas C_t and C_r for r.v.'s X^2 and Y^2 for which equality on either side of (3.1) is attained. Consequently, in order to show that the bounds in (3.2) are best-possible, we have to show that, for any such C_t or C_r, we can determine a copula $C_{X,Y}$ connecting r.v.'s X and Y such that $C_{X^2,Y^2} = C_t$ or $C_{X^2,Y^2} = C_r$, as the case may be. Letting $C = C_{X,Y}$, $C^* = C_{X^2,Y^2}$, $F = F_X$ and $G = F_Y$, it follows from (2.1) and (3.1) that C and C^* are related via

$$\begin{aligned} C^*(F(\sqrt{u}) - F(-\sqrt{u}), G(\sqrt{v}) - G(-\sqrt{v})) = {} \\ C(F(\sqrt{u}),G(\sqrt{v})) - C(F(-\sqrt{u}),G(\sqrt{v})) \\ - C(F(\sqrt{u}),G(-\sqrt{v})) + C(F(-\sqrt{u}),G(-\sqrt{v})), \end{aligned} \tag{3.3}$$

for all $u,v \geq 0$.

Thus our problem would be solved if, for any given copula C^*, we could always find a copula C such that (3.3) holds for all d.f.'s F and G and all $u,v \geq 0$. The following lemmas show that this is decidedly not the case.

LEMMA 3. If (3.3) holds for all d.f.'s F and G satisfying $F(0) = G(0) = 0$ and all $u,v \geq 0$, then $C^* = C$.

PROOF. If $F(0) = G(0) = 0$, then $F(-\sqrt{u}) = G(-\sqrt{v}) = 0$ for all $u,v \geq 0$, whence (3.3) reduces to $C^*(F(\sqrt{u}),G(\sqrt{v})) = C(F(\sqrt{u}),G(\sqrt{v}))$. Now for any $a,b \in [0,1]^2$ we may further specify F, G, u, v so that $F(\sqrt{u}) = a$ and $G(\sqrt{v}) = b$; thus $C^* = C$.

LEMMA 4. Let C^* and C be copulas. Then (3.3) holds for all d.f.'s F and G and all $u,v \geq 0$ if and only if $C^*(a,b) = C(a,b) = ab$ for all $(a,b) \in [0,1]^2$.

PROOF. By Lemma 3, we have $C^* = C$. Now let $G(0) = 0$, let F be arbitrary, and let $a = F(\sqrt{u}) - F(-\sqrt{u})$, $b = F(-\sqrt{u})$, and $c = G(\sqrt{v})$. Then (3.3) becomes

$$C(a+b,c) = C(a,c) + C(b,c) \quad \text{for } a,b \geq 0,\ a+b \leq 1,\ 0 \leq c \leq 1. \tag{3.4}$$

Letting $f_c(x) = C(x,c)$, we have that the function f_c satisfies Cauchy's equation on the restricted domain $\{(a,b) | a,b \geq 0, a + b \leq 1\}$, whence (see Daróczy and Losonczi [7]) it follows that $f_c(x) = xf_c(1) = xc$, and this completes the proof.

Lemmas 3 and 4 show that in order to have a meaningful problem it is necessary to restrict the class of d.f.'s in (3.3). We shall consider the important special case in which the d.f.'s F and G are symmetric about 0, so that $F(\sqrt{u}) = 1 - F(-\sqrt{u})$ and $G(\sqrt{v}) = 1 - G(-\sqrt{v})$. In this case, letting $a = F(\sqrt{u})$ and $b = G(\sqrt{v})$ in (3.3) leads to the problem: Given a copula C^*, find a copula C such that for all (a,b) in $[1/2,1]^2$,

$$C^*(2a-1,2b-1) = C(a,b) - C(1-a,b) - C(a,1-b) + C(1-a,1-b). \tag{3.5}$$

But this is easily done (basically because the points (a,b), $(1-a,b)$, $(a,1-b)$ and $(1-a,1-b)$ belong to different quarters of the unit square). For example, if we assign equal mass to the rectangles $[a,1] \times [b,1]$ and $[0,1-a] \times [0,1-b]$ and no mass to the squares $[0,1/2] \times [1/2,1]$ and $[1/2,1] \times [0,1/2]$, i.e., if we assume that

$$\begin{aligned} C(1-a,1-b) &= 1 - a - b + C(a,b), \\ C(1-a,b) &= 1 - a = \min(1-a,b), \\ C(a,1-b) &= 1 - b = \min(a,1-b), \end{aligned} \tag{3.6}$$

then substituting into (3.5) yields

$$C(a,b) = \frac{1}{2}\{1 + C^*(2a-1,2b-1)\}, \quad \text{for all } (a,b) \in [1/2,1]^2,$$

whence using (3.6), we have

$$C(a,b) = \begin{cases} \frac{1}{2}\{1 + C^*(2a-1,2b-1)\}, & (a,b) \in [1/2,1]^2, \\ \frac{1}{2}\{2a + 2b - C^*(1-2a,1-2b)\}, & (a,b) \in [0,1/2]^2, \\ \min(a,b), & \text{otherwise.} \end{cases} \tag{3.7}$$

A tedious but straightforward computation shows that C is indeed a copula. Thus we have proved:

THEOREM 5. Let X and Y be random variables whose distribution functions are continuous and symmetric about 0. Let C^* be any given copula. Then there exists a copula C, e.g., the copula given by (3.7), such that if $C_{X,Y} = C$, then $C_{X^2,Y^2} = C^*$.

One can restate the preceding result in the following manner: Let F and G be continuous one-dimensional d.f.'s which are symmetric about 0, and let X and Y be r.v.'s with df(X) = F and df(Y) = G. Then, for any specified two-dimensional d.f. H^* whose margins are $df(X^2)$ and $df(Y^2)$, there exists a two-dimensional d.f. H with margins F and G such that the joint d.f. of X^2 and Y^2 is H^* whenever the joint d.f. of X and Y is H.

The next result is an immediate consequence of (2.7) and (2.8) and Theorem 5.

THEOREM 6. Under the hypotheses of Theorem 5, the bounds in Theorem 1 are pointwise best-possible, that is: for any x in **R**,

(i) There exists a copula $C^t_{X,Y}$, dependent only on the value t of $\tau_W(F_{X^2},F_{Y^2})$ at x, such that

$$df(X^2+Y^2)(x) = \tau_W(F_{X^2},F_{Y^2})(x) = t,$$

(ii) There exists a copula $C^r_{X,Y}$, dependent only on the value r of $\rho_W(F_{X^2},F_{Y^2})$ at x, such that

$$df(X^2+Y^2)(x) = \rho_W(F_{X^2},F_{Y^2})(x) = r.$$

Since the d.f.'s F_X and F_Y in Theorem 6 are symmetric, we have

$$F_{X^2}(x) = \begin{cases} 2F_X(\sqrt{x}) - 1, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

and similarly for $F_{Y^2}(x)$. Thus an application of Lemma 1 and Lemma 2 immediately yields

THEOREM 7. If the r.v's X and Y are identically distributed and if their common d.f. G is symmetric about 0 and concave on $(0,\infty)$, then for all $x \geq 0$,

$$\max\{4G(\sqrt{x/2}) - 3, 0\} \leq df(X^2 + Y^2)(x) \leq 2G(\sqrt{x}) - 1. \tag{3.8}$$

These bounds are explicit, universal and pointwise best-possible.

As an immediate corollary to Theorem 7, we have best-possible bounds on the radial error $(X^2+Y^2)^{1/2}$. Since $\Pr\{(X^2+Y^2)^{1/2} < x\} = \Pr\{(X^2+Y^2) < x^2\}$, we need only replace x by x^2 to obtain

COROLLARY 8. Let X and Y be identically distributed r.v.'s whose common d.f. G is symmetric about 0 and concave on $(0,\infty)$. Then for all $x \geq 0$,

$$\max\{4G(x/\sqrt{2}) - 3, 0\} \leq df\{(X^2 + Y^2)^{1/2}\}(x) \leq 2G(x) - 1. \tag{3.9}$$

We conclude this section with several remarks.

1. The copula C in (3.7) is not the only solution of (3.5). For example, if in (3.6) we set

$$C(1 - a,b) = b - C(a,b),$$

and

$$C(a,1 - b) = a - C(a,b),$$

i.e., if we assign equal mass to each of the four rectangles $[a,1] \times [b,1]$, $[0,1-a] \times [0,1-b]$, $[a,1] \times [0,1-b]$, and $[0,1 - a] \times [b,1]$, then we find that

$$C(a,b) = \begin{cases} \frac{1}{4}\{2a + 2b - 1 + C^*(2a - 1, 2b - 1)\}, & (a,b) \in [1/2,1]^2, \\ \frac{1}{4}\{2a + 2b - 1 - C^*(1 - 2a, 2b - 1)\}, & (a,b) \in [0,1/2) \times [1/2,1], \\ \frac{1}{4}\{2a + 2b - 1 - C^*(2a - 1, 1 - 2b)\}, & (a,b) \in [1/2,1] \times [0,1/2), \\ \frac{1}{4}\{2a + 2b - 1 + C^*(1 - 2a, 1 - 2b)\}, & (a,b) \in [0,1/2)^2. \end{cases} \tag{3.10}$$

This choice is interesting because any two r.v.'s whose copula is given by (3.10) are uncorrelated in the sense of Spearman's ρ, Kendall's τ, Blomqvist's medial correlation coefficient, and Pearson's r (provided the requisite first and second moments exist). The proofs of these facts are straightforward calculations using the expressions given in [8] and [9] for these measures.

2. If C is a copula and is C^* is defined via (3.5) for (a,b) in $[1/2,1]^2$, then C^* is also a copula. Letting $\alpha = 2a - 1$ and $\beta = 2b - 1$, this fact can be expressed in the following more pleasing form:

THEOREM 9. If C is a copula and C^* is defined by

$$\begin{aligned} C^*(\alpha,\beta) = C((1 + \alpha)/2,(1 + \beta)/2) - C((1 - \alpha)/2,(1 + \beta)/2) \\ - C((1 + \alpha)/2,(1 - \beta)/2) + C((1 - \alpha)/2,(1 - \beta)/2) \end{aligned}$$

for all (α,β) in $[0,1]^2$, then C^* is a copula.

Note that $C^*(\alpha,\beta)$ is the mass that C assigns to the rectangle $[(1 + \alpha)/2,(1 + \beta)/2] \times [(1 + \alpha)/2,(1 + \beta)/2]$ which is centered at the point $(1/2,1/2)$.

3. Since the copula of any pair of r.v.'s X and Y is invariant under strictly increasing transformations f and g of X and Y, respectively, we have

$$C_{X^2,Y^2} = C_{|X|,|Y|} = C_{f(|X|),g(|Y|)}.$$

4. For completeness sake we note that if the r.v.'s X and Y are non-negative, then in view of Lemma 3, the problem of showing that the bounds in (3.2) are pointwise best-possible is (trivially) solved by choosing $C_{X,Y} = C_t$ and $C_{X,Y} = C_r$, respectively.

4. AN EXAMPLE.

Suppose X and Y are normally distributed with mean 0 and variance σ^2. If we let Φ denote the distribution function of the standard normal, then since $\Phi(x/\sigma)$ clearly satisfies the hypotheses of Corollary 8, we immediately have

$$\max\{4\Phi(x/\sigma\sqrt{2}\,) - 3,0\} \le \mathrm{df}\{(X^2 + Y^2)^{1/2}\}(x) \le 2\Phi(x/\sigma) - 1.$$

These bounds are graphed in Figure 1, along with the d.f. P(x) for the radial error from three other bivariate distributions which have $N(0,\sigma^2)$ marginals. The two labelled $r = 0$ and $r = \pm 1$ are bivariate normals with the indicated correlation coefficient, for which $P(x) = 1 - \exp(-x^2/2\sigma\sqrt{2}\,)$ and $P(x) = 2\Phi(x/\sigma\sqrt{2}\,) - 1$, respectively. The third is *Vaswani's bivariate normal* [10], for which the copula is C(a,b) from (3.7) with $C^*(a,b) = W(a,b)$ from (2.4).

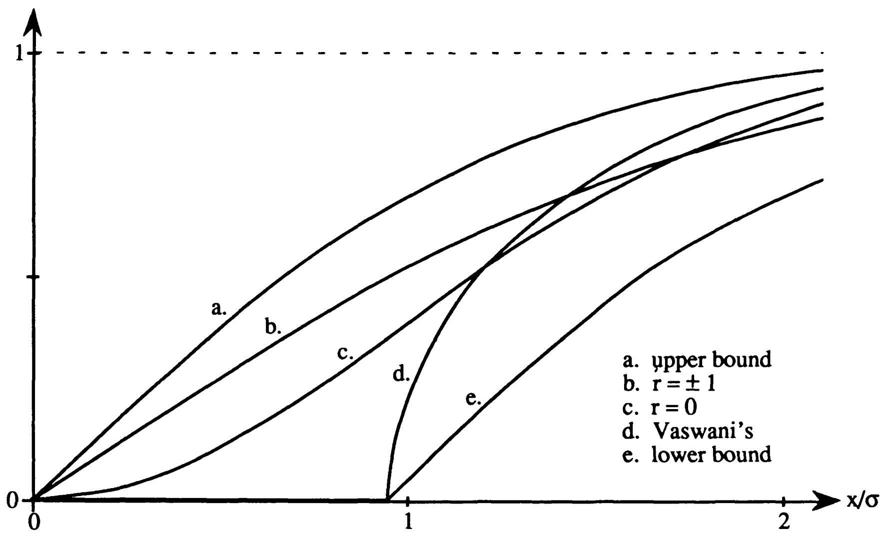

Figure 1. Distribution functions for radial error and the bounds in the normal case.

In this example we can compute explicit bounds for the *circular probable error* (C.E.P.), the two-dimensional analog of the probable error of a single random variable, which is defined as the radius of the mean-centered circle encompassing 50% of the probability mass, or the median of the distribution of $(X^2+Y^2)^{1/2}$. The bounds are x_L and x_U, where $2\Phi(x_L/\sigma) - 1 = 1/2$ and $\max\{4\Phi(x_U/\sigma\sqrt{2}\,) - 3,0\} = 1/2$, so that

$$x_L = \sigma\Phi^{-1}(3/4) \cong 0.67449\sigma,$$
$$x_U = \sigma\sqrt{2}\,\Phi^{-1}(7/8) \cong 1.62684\sigma.$$

As a point of comparison, the C.E.P. for independent $N(0,\sigma^2)$ random variables is $\sigma\sqrt{2\ln 2} \cong 1.17741\sigma$.

As a consequence of Theorems 5 and 6, we can construct a bivariate distribution with normal marginals which has minimum (or maximum) C.E.P. We construct a copula for such a distribution using (3.7) where C^* is taken to be C_r from (2.10) with $r = 1/2$; that is,

$$C(a,b) = \begin{cases} \max(a+b-1/2,\ 1/4), & (a,b) \in [1/4,1/2)^2, \\ \max(a+b-3/4,\ 1/2), & (a,b) \in [1/2,3/4]^2, \\ \min(a,b), & \text{otherwise.} \end{cases}$$

When this copula is endowed with $N(0,\sigma^2)$ margins, we obtain the bivariate distribution function

$$H_{X,Y}(x,y) = \begin{cases} \max\{\Phi(x/\sigma) + \Phi(y/\sigma) - 1/2\}, & (x,y) \in [\sigma\Phi^{-1}(1/4),0)^2, \\ \max\{\Phi(x/\sigma) + \Phi(y/\sigma) - 3/4\}, & (x,y) \in [0,\sigma\Phi^{-1}(3/4)]^2, \\ \min\{\Phi(x/\sigma), \Phi(y/\sigma)\}, & \text{otherwise.} \end{cases}$$

This distribution is singular, with half the probability mass concentrated on segments of the line $x = y$ for $x < \sigma\Phi^{-1}(1/4)$ and $x > \sigma\Phi^{-1}(3/4)$, and half the mass on the two branches of the curve $\Gamma: |\Phi(x/\sigma)+\Phi(y/\sigma)-1| = 1/4$. But Γ lies entirely within the origin-centered circle of radius $\sigma\Phi^{-1}(3/4)$ and the line segments lie outside this circle; thus this distribution attains minimum C.E.P.

We note here that there is the following probabilistic interpretation for this distribution: Let T be a uniform (0,1) random variable, and set

$$X = \sigma\Phi^{-1}(T),$$

$$Y = \begin{cases} \sigma\Phi^{-1}(3/4 - T), & T \in [1/4,1/2), \\ \sigma\Phi^{-1}(5/4 - T), & T \in [1/2,3/4], \\ X, & \text{otherwise.} \end{cases}$$

Then (X,Y) has the bivariate d.f. $H_{X,Y}$ given above.

To construct a bivariate distribution with maximum C.E.P., we proceed in a similar fashion beginning with C_t from (2.9) with $t = 1/2$. This yields the d.f.

$$H_{X,Y}(x,y) = \begin{cases} \max\{\Phi(x/\sigma) + \Phi(y/\sigma) - 1,\ 3/4\}, & x,y > \sigma\Phi^{-1}(3/4), \\ \max\{\Phi(x/\sigma) + \Phi(y/\sigma) - 1/4,\ 0\}, & x,y < \sigma\Phi^{-1}(1/4), \\ \min\{\Phi(x/\sigma), \Phi(y/\sigma)\}, & \text{otherwise.} \end{cases}$$

It can be shown that this distribution places exactly half the probability mass within the circle centered at the origin with radius $\sigma\sqrt{2}\,\Phi^{-1}(7/8)$, so that this distribution has maximum C.E.P.

5. EXTENSIONS.

Our results extend naturally to n dimensions and to arbitrary strictly increasing functions on $[0,\infty)$. For example, if we replace the sum of squares in two dimensions by the sum of p^{th} powers in n dimensions, then, for $x \geq 0$, the methods employed in Theorem 7 (with the r.v.'s X and Y replaced by $X_1,\cdots,X_n$) give,

$$\max\{2nG([x/n]^{1/p}) - (2n-1),0\} \leq df(|X_1|^p + \cdots + |X_n|^p)(x) \leq 2G(x^{1/p}) - 1.$$

For the analog of the radial error, namely the l_p norm $(|X_1|^p + \cdots + |X_n|^p)^{1/p}$ in n dimensions, Corollary 8 yields

$$\max\{2nG(x/n^{1/p}) - (2n-1),0\} \leq df\{(|X_1|^p + \cdots + |X_n|^p)^{1/p}\}(x) \leq 2G(x) - 1.$$

It is worth noting that the upper bound $2G(x) - 1$ is independent of both dimension and norm: this is a consequence of the idempotency of ρ_W exhibited in Lemma 1.

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