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A PROBLEM OF THERMAL SHOCK WITH THERMAL RELAXATION

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ABSTRACT. The problem of a semi-infinite medium subjected to thermal shock on its plane boundary is solved using the generalized theory of thermoelasticity. The expressions for temperature, strain and stress are presented. The results are exhibited graphically and compared with previous results.

KEY WORDS AND PHRASES. Generalized theory, relaxation time, Lame's elastic constants, diffusion equation, relaxation constant.
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1. INTRODUCTION.

Thermoelastic problems are solved using the governing dynamical equations for the displacement and temperature. These equations are two partial differential equations.

Equation of motion:

$$\rho \ddot{u}_i = (\lambda + \mu)\, u_{j,ij} + \mu u_{i,jj} - (3\lambda + 2\mu)\alpha T_{,i}\,. \tag{1.1}$$

Equation of energy:

$$k\, T_{,ii} = \rho\, C_e \dot{T} + (3\lambda + 2\mu)\, \alpha\, T_o\, \dot{\varepsilon}_{kk}. \tag{1.2}$$

The first equation is of wave type and the other is of diffusion type. For an isotropic, homogeneous elastic body subjected to a shock, the latter equation shows that the disturbance will be felt instantaneously at distances far from its source. As the equations are coupled this effect will be felt in both temperature and displacement. Such a behavior is physically inadmissible and contradicts the existing theory of heat conduction.

Many researchers, for example Morse and Feshbach [1], Boley [2], Baumister and Hamill [3] have discussed this paradox and suggested some modifications in the governing equations. Lord and Shulman [4] proposed the Generalized theory of thermoelasticity, where the time lag needed for the onset of thermal wave-relaxation time is considered and is well-known as the modified coupled heat conduction equation. Based on this modified theory, considerable work is being done by many authors (Fox [5], Ignaczak [6], Sherief and Anwar [7], and Choudhuri and Sain [8]).

Here the problem of an isotropic, homogeneous half-space subjected to thermal shock on its plane boundary is solved using the Laplace transform technique. The equations concerning the generalized theory of thermoelasticity are used to solve the said problem. The boundary condition for temperature is in the form of exponential heating, a more realistic situation. After effecting the Laplace inversion, the expressions for temperature, strain and stress are obtained. As a special case, the results due to Danilovskaya [9] for step-type boundary conditions and that of Sternburg et al [10] for ramp type boundary condition can be obtained. Further by setting relaxation constant to zero, the results due to Daimaruya et al [11] are obtained.

2. FORMULATION OF THE PROBLEM.

Consider an isotropic homogeneous half space, subjected to a thermal disturbance on its boundary. The governing equations of the generalized theory of thermoelasticity for the one dimensional case, are

$$k \frac{\partial^2 T}{\partial x^2} = \rho C_e \left(\frac{\partial T}{\partial t} + \tau_o \frac{\partial^2 T}{\partial t^2}\right) + (3\lambda + 2\mu)\, \alpha T_o \left(\frac{\partial^2 u}{\partial x \partial t} + \tau_o \frac{\partial^3 u}{\partial x \partial t^2}\right) \tag{2.1}$$

$$\sigma_{xx} = (\lambda + 2\mu) \frac{\partial u}{\partial x} - (3\lambda + 2\mu)\, \alpha\, (T - T_o) \tag{2.2}$$

$$(\lambda + 2\mu) \frac{\partial^2 u}{\partial x^2} - (3\lambda + 2\mu)\, \alpha \frac{\partial T}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2} \tag{2.3}$$

where k, ρ, C_e, α, τ_o are thermal conductivity, density, specific heat, coefficient of linear thermal expansion and the relaxation time, respectively. λ and μ are the well known Lame's elastic constants. Here T, σ_{xx}, and u are temperature, stress and displacement, respectively.

The initial and boundary conditions are

$$T(x,0) = T_o,\ x > 0$$

$$u(x,0) = 0 = \left(\frac{\partial u}{\partial t}\right)_{t=0},\ \left(\frac{\partial T}{\partial t}\right)_{t=0} = 0 \tag{2.4}$$

$$T(0,t) = T_o\, [1 - \exp(-t/t_o)]$$

$$u'(0,t) = \frac{(3\lambda + 2\mu)\alpha T_o}{(\lambda + 2\mu)}\, [1 - \exp(-t/t_o)]. \tag{2.5}$$

In the above, T_o, and t_o are constants. The step type boundary condition is obtained when $t_o = 0$, i.e. $T(0,t) = H(t)$, and the ramp type boundary condition is obtained by expanding the exponential type and neglecting the higher order terms. Here H(t) is Heaviside unit step function.

The regularity boundary conditions are

$$T(x,t),\ u(x,t),\ \sigma_{xx}(x,t) \to 0 \text{ as } x \to \infty. \tag{2.6}$$

Introducing the following non-dimensional variables

$$z = \left(\frac{\lambda + 2\mu}{\rho}\right)^{1/2} \frac{\rho C_e}{k} x, \quad y = \frac{\lambda + 2\mu}{\rho} \frac{\rho C_e}{k} t,$$

$$\theta = \frac{T - T_o}{T_o}, \quad \Sigma = \sigma_{xx} (3\lambda + 2\mu) \alpha T_o \tag{2.7}$$

$$U = \left[\rho \left(\frac{\lambda + 2\mu}{\rho}\right)^{3/2} \frac{1}{(3\lambda + 2\mu)\alpha T_o} \frac{\rho C_e}{k}\right] u$$

for distance, time, temperature, stress, and displacement respectively in equations (2.1) - (2.3), we get

$$\Theta'' - \Theta' - \beta\ddot{\Theta} = \bar{e} (\dot{U}' + \beta\ddot{U}'),$$

$$U'' - \Theta' = \ddot{U}, \tag{2.8abc}$$

$$\Sigma = U' - \Theta$$

where 'dot' and 'dash' denote differentiation with respect to y and z respectively and

$$\beta = \left(\frac{\lambda + 2\mu}{\rho}\right) \frac{\rho C_e}{k} . \tau_o, \quad \bar{e} = \frac{(3\lambda + 2\mu)^2 \alpha^2 T_o}{(\lambda + 2\mu) \rho C_e} .$$

Here β and $\bar{e}$ are the relaxation constant and thermoelastic coupling constant, respectively.

In (2.8a), the strain acceleration term $(\ddot{U}')$ can be ignored as the product $\bar{e}.\beta$ is much less than $\bar{e}$ or β in the intermediate and room temperature.

Now our initial and boundary conditions (2.4) - (2.6) reduce to

$$\Theta(z,0) = U(z,0) = \left(\frac{\partial U}{\partial y}\right)_{y=0} = 0 = \left(\frac{\partial \Theta}{\partial y}\right)_{y=0},$$

$$\Theta(0,y) = - \exp(-y/\tau'_o) \tag{2.9}$$

$$U'(0,y) = 1 - \exp(-y/\tau'_o)$$

$$\Theta(z,y), U(z,y), \Sigma(z,y) \to 0 \quad \text{as } z \to \infty$$

where

$$\tau'_o = \left(\frac{\lambda + 2\mu}{\rho}\right) \frac{\rho C_e}{k} t_o .$$

3. SOLUTION OF THE PROBLEM.

In order to obtain the solutions for the temperature (Θ) and strains (U'), we first eliminate U,Θ from the first two equations of (2.8), namely

$$\Theta'''' - (1 + \beta) \ddot{\Theta}'' - (1 + \bar{e}) \dot{\Theta}'' + \dddot{\Theta} + \beta \ddddot{\Theta} = 0 \tag{3.1}$$

$$U'''' - (1 + \beta) \ddot{U}'' - (1 + \bar{e}) \dot{U}'' + \dddot{U} + \beta \ddddot{U} = 0 . \tag{3.2}$$

Applying the Laplace transform to (3.1), (3.2) and (2.9), one obtains

$$\bar{\Theta}'''' - p[p(1+\beta) + (1+\bar{e})]\, \bar{\Theta}'' + p^2(p+\beta p^2)\, \bar{\Theta} = 0$$

$$\bar{U}'''' - p[p(1+\beta) + (1+\bar{e})]\, \bar{U}'' + p^2(p+\beta p^2)\, \bar{U} = 0. \qquad (3.3)$$

Here $\bar{\Theta}$, $\bar{U}$ are Laplace transforms of Θ and U respectively and p is the transform parameter, and

$$\bar{\Theta}(0,p) = -\tau_o'/p_o$$

$$\bar{U}'(0,p) = 1/pp_o \qquad (3.4)$$

$$\bar{\Theta}(z,0) = \bar{U}(z,0) = 0.$$

In the above,

$$p_o = p\,\tau_o' + 1.$$

From (3.3) and (3.4), we get

$$\bar{\Theta} = A_1 \exp(-\alpha_2 z) + A_2 \exp(-\alpha_2 z)$$

$$\bar{U} = B_1 \exp(-\alpha_1 z) + B_2 \exp(-\alpha_2 z) \qquad (3.5)$$

where

$$A_{1,2} = \pm D\, [\bar{e} + \tau_o'(\alpha_{2,1}^2 - pp_o')]$$

$$\alpha_{1,2}\, B_{1,2} = \mp D[\bar{e} - \tau_o' pp_o' + p - \alpha_{2,1}^2/p]$$

$$\frac{1}{D} = p_o(\alpha_1^2 - \alpha_2^2)$$

$$p_o' = 1 + \beta p$$

$$\alpha_{1,2}^2 = \frac{p}{2}[(1+\beta)p + (1+\bar{e}) \pm [((1+\beta)p + (1+\bar{e}))^2 - 4pp_o']^{1/2}].$$

We consider a special case in which the relaxation constant (β) is expressed in terms of the coupling constant $\bar{e}$, i.e. $\beta = \dfrac{1}{1+\bar{e}}$. For this value of β, we get

$$\alpha_1^2 = p^2 + p/\beta, \quad \alpha_2^2 = \beta p^2 \qquad (3.6)$$

and then the transformed temperature and strain become

$$\bar{\Theta}(z,p) = N(p)\, [(\bar{e}-\tau_o' p)\exp(-\alpha_1 z)-(p^2\tau_o'(1-\beta)+\bar{e}p_o)\exp(-\alpha_2 z)] \qquad (3.7)$$

$$\bar{U}'(z,p) = N(p)[[\bar{e} + p(1-\beta-\tau_o') - \beta_o\tau' p^2]\exp(-\alpha_1 z)$$

$$+ (1 + \tau_o' p_o' p)\exp(-\alpha_2 z)] \qquad (3.8)$$

where

$$N(p) = \frac{1}{pp_o(p + b^2)\ (1-\beta)}$$

and

$$b^2 = \frac{1}{\beta(1 - \beta)} .$$

Inverting (3.7) and (3.8), we get

$$\Theta = C_1 I(0) + C_2 \exp(-y/\tau_o') \ I(1/\tau_o') + C_3 \exp(-b^2 y) I(b^2) + C_4\ \phi_1(1/\tau_o') + C_5\ \phi_1(b^2) + C_6 \phi_2(1/\tau_o') + C_7 \phi_2(b^2) \tag{3.9}$$

and

$$U' = K_1 I(\phi) + K_2 I(1/\tau_o') + K_3 I(b^2) + K_4 \phi_1(1/\tau_o') + K_5\ \phi_1(b^2) + K_6 \phi(1/\tau_o') + K_7 \phi_2(b^2) - (\beta/(1-\beta))(e^{-z/2\beta} - 1) \tag{3.10}$$

where

$$I(w) = \int_o^y z \exp[(w - \frac{1}{2\beta})\ell]\ F(z,\ell)\ H(\ell - z) d\ell$$

$$C_1 = (1 - \beta)\ D_1 , \quad C_2 = b^2 \tau_o'\ D_1 D_2$$

$$C_3 = -\beta(b^2\ \tau_o' + \bar{e})\ D_1 D_2, \quad C_4 = -C_2/D_1$$

$$C_5 = -C_3/D_1, \quad C_6 = D_2, \quad C_7 = -D_3 . D_2/(1-\beta),$$

$$K_1 = C_1, \quad K_2 = b^2(\tau_o' - \beta)\ D_1 D_2 \exp(-y/\tau_o')$$

$$K_3 = D_1 D_2 D_3 b^2 \exp\ (-b^2 y)/\ (1-\beta), \quad K_4 = \frac{-b^2(\tau_o' - \beta)}{\tau_o'} D_2 \exp(-z/2\beta)$$

$$K_5 = b^2 \beta\ D_2 D_3 \exp\ (-z/2\beta), \quad K_6 = -D_2/(b^2 \tau_o'\ (1 - \beta)^2)$$

$$K_7 = b^2 \beta\ D_2 D_3/\ (1 - \beta)^2, \quad D_1 = 1/(144\ \beta^2)$$

$$D_2 = 1/(1 - b^2 \tau_o'), \quad D_3 = (1-\beta)^2 + \tau_o'$$

$$\phi_1(w) = [1 - \exp[(z-y)w]]\ H(y-z)/w, \quad \phi_2(w) = [1 - \exp[(\sqrt{\beta z} - y)w]]\ H(y - \sqrt{\beta z})/w$$

$$F(z,y) = I_1(Z'/2\beta)/D_1 Z'$$

$$H(y-z) = 0, \quad 0 < y < z$$

$$= 1, \quad y > z$$

and

$$Z' = (y^2 - z^2)^{1/2}$$

Here I_1 is the modified Bessel function of first kind.

The expression for stress can be obtained from (2.8c), (3.9) and (3.10). Taking $\tau_o' = 0$, recovers the results due to Danilovskaya [9] and setting $\beta = 0$, recovers the results of Daimaruya [11].

4. RESULTS AND DISCUSSION.

The results for temperature, strain and stress distributions are evaluated numerically and exhibited graphically in figures 1 to 3, for a particular value of the relaxation constant (β) given by $\beta = 1/(1+\bar{e})$. The transport of thermal energy in the medium, i.e. either a diffusion process or a wave Like process depends on the magnitude of the relaxation constant. It was observed that at low temperatures the magnitude of the relaxation constant becomes significant and the energy equation predicts a wave-type phenomenon. The magnitudes of the coupling and relaxation constants were calculated over a range of intermediate and high temperatures by Lord [12]. The values of the coupling paramter $\bar{e}$ are smaller than unity for most of the materials.

Here for the computation, the relaxation constant (β) was taken as 0.98 and 0.76 (the corresponding values of $\bar{e}$ are 0.1 and 0.31 respectively) and the values 0.25, 0.5, 1 and 2 for the τ_o'. The time-dependence of the non-dimensional temperature (Θ), strain (U'), and stress (Σ) are depicted as a function of non-dimensional time y at the non-dimensional distance z = 2 for $\tau_o' = 0.5$.

As the relaxation constant increases the corresponding components of temperature, strain and stress decrease. The gap between these corresponding parameters increases, due to the effect of relaxation time unlike that of coupled theory. It may be mentioned that a similar phenomena was observed by Daimaruya [11] in case of coupled theory. Moreover due to presence of Heaviside unit step function, the two discontinuities can occur in temperature and stress at the wavefronts $y = z$ and $y = \sqrt{\beta}\, z$ and the corresponding accoustic velocity (v_1) and thermal velocity (v_2) at the wavefronts are 1 and $1/\sqrt{\beta}$ respectively. Last but not least, the results obtained here which including the effect of the relaxation time are more general.

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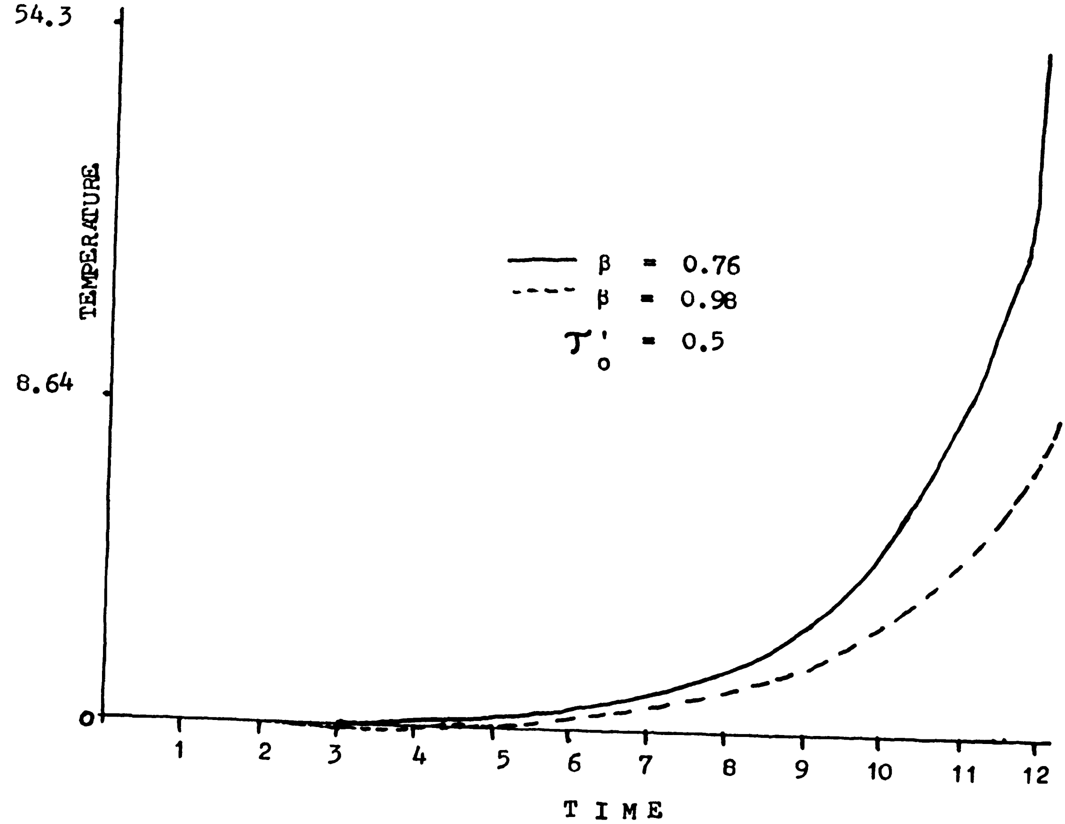

Fig. 1. Time dependence of temperature at z = 2.

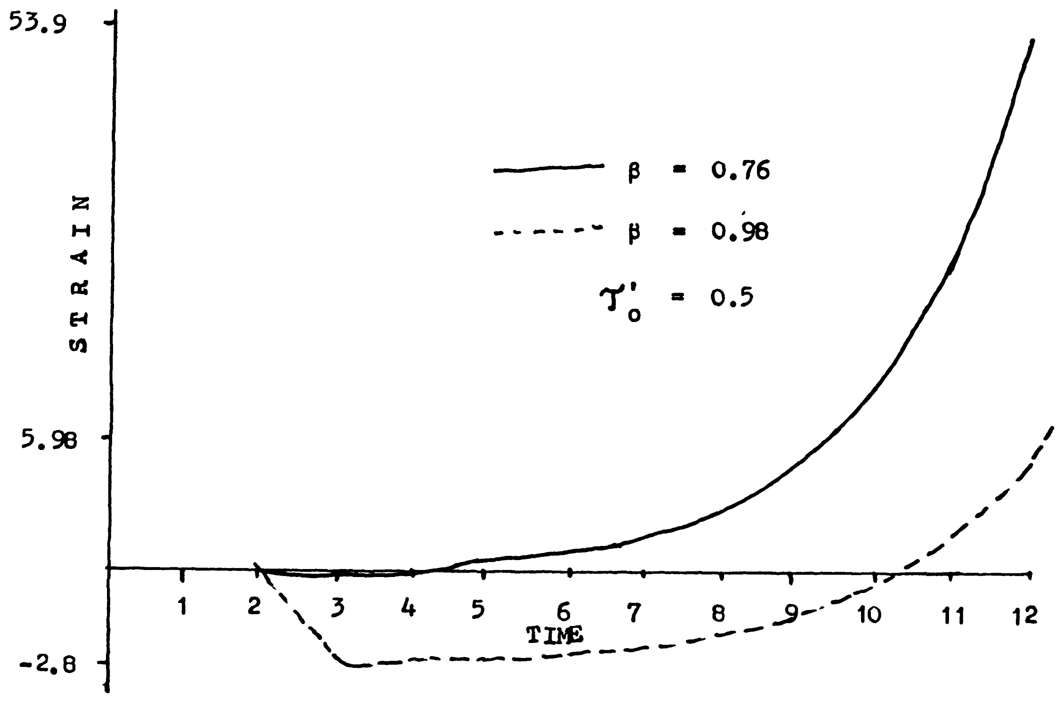

Fig. 2. Time dependence of strain at z = 2.

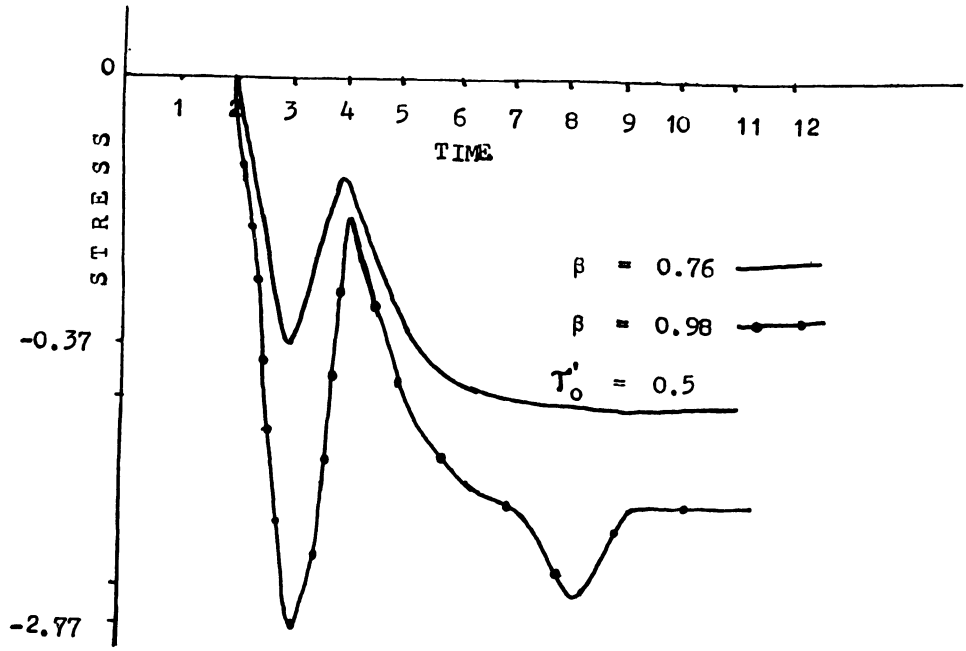

Fig. 3. Time dependence of stress at z = 2.

REFERENCES

1. MORSE, P.M. and FESHBACH, H., Methods of Theoretical Physics, McGraw-Hill, New York, 1953.
2. BOLEY, B.A., The Analysis of Problems of Heat Conduction, Pergamon Press, New York, 1964.
3. BAUMISTER, K.J. and HAMILL, T.D., Hyperbolic Heat Conduction Equation, Jnl. of Heat Transfer, ASME Trans. 91 (1969), 543-548.
4. LORD, H.W. and SHULMAN, Y., A Generalized Dynamical Theory of Thermoelasticity, J. Mech. Phys. Solids 15 (1967), 299-309.
5. FOX, N., Generalized Thermoelasticity, Int. J. Engg. Sci. 7 (1969), 437-445.
6. IGNACZAK, J., A strong discontinuity wave in Thermoelasticity with Relaxation Times, Jnl. of Thermal stresses 8 (1985), 25-40.
7. SHERIEF, H.H. and ANWAR, N., Problem in Generalized Thermoelasticity., Jnl. of Thermal Stresses 9 (1986), 165-181.
8. ROY CHOUDHURI, S.K. and SAIN, G., Instantaneous Heat Sources...With Thermal relaxation., Ind. J. Pure. Appl. Math. 13 (1982), 1340-1353.
9. DANILOVSKAYA, V.I., Thermal Stresses in an Elastic Half-space....Prikl. Mat. Mech. 14 (1950), 816-820.
10. STERNBERG, E. and CHAKRAVORTHY, J.G., On Inertia Effects in a Transient Thermoelastic Problem, Trans. ASME. Ser. E. 26 (1959), 503.
11. DAIMARUYA, M. and NATTO, N., A Transient Coupled Thermoelastic Problem in the Semi-infinite Medium. Bull. JSME. 16 (1973), 1494-1505.
12. LORD, H.W., Ph.D. Dissertation, North Western University, Evanstan, 1966.