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A COMMON FIXED POINT THEOREM FOR TWO SEQUENCES OF SELF-MAPPINGS

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ABSTRACT. In this paper a common fixed point theorem for two sequences of self-mappings from a complete metric space M to M is proved. Our theorem is a generalization of Hadzic's fixed point theorem[1].

KEY WORDS AND PHRASES. A common fixed point, self-mappings and complete metric spaces.

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1. INTRODUCTION.

Banach's fixed point theorem has been generalized by many authors. Among such investigations there are several, interesting and important studies[2]. Particularly, K. Iseki[3] proved a fixed point theorem of a sequence of self-mappings from a complete metric space M to M. We are interested in fixed point theorems of a sequence of self-mappings since they pertain to the problem of finding an equilibrium point of a difference equation $x_{n+1} = f(n, x_n)$ $(n = 1,2,...)$.

Recently O. Hadzic proved the existence of a common fixed point for the sequence of self-mappings $\{A_j\}(j = 1,2,...)$, S and T where A_j commutes with S and T. His result is as follows:

THEOREM 1. Let (M,d) be a complete metric space, S,T:M $\rightarrow$ M be continuous, $A_j: M \rightarrow SM \cap TM (j = 1,2,...)$ so that A_j commutes with S and T and for every $i,j (i = = j, i,j = 1,2,...)$ and every $x,y \in M$:

$$d(A_i x, A_j y) \le q d(Sx, Ty), \quad 0 < q < 1 \tag{1.1}$$

Using Theorem 1, he gave a generalization of Gohde's fixed point theorem and extended Krasnoseliski's fixed point theorem.

In this paper we shall present a generalization of Hadzic's fixed point theorem.

2. MAIN THEOREMS.

Let N denote the set of all positive integers. In this section we shall prove the following theorem.

THEOREM A. Let (M,d) be a complete metric space and let $\{A_p\}, \{B_q\}(p, q = 1, 2, ...)$, be two sequences of mappings from M to M.

Suppose that the following conditions are satisfied; for all $m, n \in N$ and all $x, y \in M$,

(a) there exists a constant k $(0 < k < 1)$ such that

$$d(A_{2n-1}x, A_{2n}y) \leq kd(B_{2n-1}x, B_{2n}y),$$

$$d(A_{2n}x, A_{2m+1}y) \leq kd(B_{2n}x, B_{2m+1}y), \text{ for all } m \geq n \geq 1,$$

(b) $A_{2n}B_{2m} = B_{2m}A_{2n}$ and $A_{2n-1}B_{2m-1} = B_{2m-1}A_{2n-1}$,

(c) $B_{2n}B_{2m} = B_{2m}B_{2n}$ and $B_{2m-1}B_{2n-1} = B_{2n-1}B_{2m-1}$,

(d) $A_{2n-1}(M) \subset B_{2n}(M)$ and $A_{2n}(M) \subset B_{2n+1}(M)$.

If each $B_q(q = 1, 2, ...)$ is continuous, then there exists a unique fixed point for two sequences $\{A_p\}$ and $\{B_q\}(p, q = 1, 2, ...)$.

PROOF. Let x_0 be an arbitrary point in M. By condition (d) there exists a point $x_1 \in M$ such that $A_1x_0 = B_2x_1$. Next we choose a point $x_2 \in M$ such that $A_2x_1 = B_3x_2$. Inductively, we can define by condition (d), the sequence $\{x_n\}$ such that

$$A_{2n-1}x_{2n-2} = B_{2n}x_{2n-1} \text{ and } A_{2n}x_{2n-1} = B_{2n+1}x_{2n}, \quad n \in N. \tag{2.1}$$

First of all we shall show that $\{B_nx_{n-1}\}$ is a Cauchy sequence. By (2.1) and condition (a), we obtain that for all $n \in N$

$$d(B_{2n-1}x_{2n-2}, B_{2n}x_{2n-1}) = d(A_{2n-2}x_{2n-3}, A_{2n-1}x_{2n-2})$$

$$\leq kd(B_{2n-2}x_{2n-3}, B_{2n-1}x_{2n-2}) = kd(A_{2n-3}x_{2n-4}, A_{2n-2}x_{2n-3})$$

$$\leq k^2d(B_{2n-3}x_{2n-4}, B_{2n-2}x_{2n-3}) \leq \cdots \leq k^{2n-2}d(B_1x_0, B_2x_1)$$

and similarly that

$$d(B_{2n}x_{2n-1}, B_{2n+1}x_{2n}) = d(A_{2n-1}x_{2n-2}, A_{2n}x_{2n-1})$$

$$\leq kd(B_{2n-1}x_{2n-2}, B_{2n}x_{2n-1}) \leq \cdots \leq k^{2n-1}d(B_1x_0, B_2x_1).$$

Since $0 < k < 1$, this implies that the sequence $\{B_n x_{n-1}\}$ is a Cauchy sequence. Thus $\{B_n x_{n-1}\}$ converges to some point v in M because M is complete. Now since each $B_q (q \in N)$ is continuous, we obtain that

$$B_{2m} v = B_{2m}(\lim_{n \to \infty} B_{2n+1} x_{2n}) = \lim_{n \to \infty}(B_{2m} B_{2n+1} x_{2n})$$

$$= \lim_{n \to \infty}(B_{2m} A_{2n} x_{2n-1}) = \lim_{n \to \infty}(A_{2n} B_{2m} x_{2n-1})$$

and similarly that $B_{2m+1} v = \lim_{n \to \infty}(A_{2n+1} B_{2m+1} x_{2n})$ and $B_{2m-1} v = \lim_{n \to \infty}(A_{2n-1} B_{2m-1} x_{2n-2})$. Hence by condition (c), we have

$$d(B_{2m} v, B_{2m+1} v) = \lim_{n \to \infty} \text{d}\ (A_{2n} B_{2m} x_{2n-1}, A_{2n+1} B_{2m+1} x_{2n})$$

$$\leq \lim_{n \to \infty} \text{kd}\ (B_{2n} B_{2m} x_{2n-1}, B_{2n+1} B_{2m+1} x_{2n})$$

$$= kd(B_{2m} v, B_{2m+1} v)$$

and $d(B_{2m} v, B_{2m-1} v) \leq kd(B_{2m} v, B_{2m-1} v)$ $(m \in N)$ in like manner, which implies that $B_m v = B_{m+1} v$ for all $m \geq 1$. Next we shall show that $A_n v = B_n v$ for all $n \leq 1$. By (2.1), conditions (b) and (c), we have

$$d(B_{2n+1} B_{2m+2} x_{2m+1}, A_{2n} v) = d(A_{2m+1} B_{2n+1} x_{2m}, A_{2n} v)$$

$$\leq kd(B_{2m+1} B_{2n+1} x_{2m}, B_{2n} v)$$

$$= kd(B_{2n+1} B_{2m+1} x_{2m}, B_{2n} v)$$

Thus letting $m \to \infty$, we obtain that $d(B_{2n+1} v, A_{2n} v) \leq kd(B_{2n+1} v, B_{2n} v)$ from which it follows that $A_{2n} v = B_{2n+1} v$ for all $n \geq 1$. And since

$$d(A_{2n-1} v, A_{2n} v) \leq kd(B_{2n-1} v, B_{2n} v) \text{ and } d(A_{2n+1} v, A_{2n} v) \leq kd(B_{2n+1} v, B_{2n} v),$$

we obtain that $A_n v = A_{n+1} v = B_{n+1} v = B_n v$ for all $n \in N$. Furthermore, for all $n \in N$, we obtain

$$d(A_{2n} v, A_{2n-1} A_{2n+1} v) \leq kd(B_{2n} v, B_{2n-1} A_{2n+1} v) = kd(A_{2n} v, A_{2n-1} A_{2n+1} v)$$

$$\text{and } d(A_{2n-1} v, A_{2n} A_{2n+1} v) \leq kd(B_{2n-1} v, B_{2n} A_{2n+1} v) = kd(A_{2n-1} v, A_{2n} A_{2n+1} v).$$

Therefore we obtain $u = A_p(u) = B_p(u)$ for all $p \geq 1$ setting $u = A_n v$ because $0 < k < 1$.

Now we shall prove that u is a unique common fixed point of $\{A_p\}$ and $\{B_p\}$. If there exists another point w such that $w = A_p w = B_p w$ for all $p > 1$, then

$$d(u,w) = d(A_{2m-1}u, A_{2m}w) \leq kd(B_{2m-1}u, B_{2m}w)$$

$$\leq kd(u,w),$$

which is a contradiction since $0 < k < 1$. Therefore u is a unique common fixed point of two sequences of self-mappings $\{A_n\}$ and $\{B_n\}$. This completes the proof.

If $S = B_{2n-1}$ and $T = B_{2n}(n = 1,2,...)$, we obtain Theorem 1 as the corollary of Theorem A. Next we obtain the following theorem which is a generalization of Theorem 1 in [4].

THEOREM B. Let (M,d) be a complete metric space and let $\{T_p\}$ $(p = 1,2,...)$ be a sequence of mappings from M to M. Suppose that the following conditions as satisfied for all $m \geq n \geq 1$ and $x, y \in M$

(e) there exists a constant h $(h > 1)$ such that

$$d(T_{2n-1}x, T_{2n}y) \geq hd(x,y) \text{ and } d(T_{2n}x, T_{2m+1}y) \geq hd(x,y),$$

(f) $T_p T_q = T_q T_p$ (p,q are even or odd respectively).

If every T_n is continuous on M and $T_n(M) = M(n = 1,2,...)$, then there exists a unique fixed point for T_n.

PROOF. Set $A_n = I$ (I is the identify map from M to M) in Theorem A. The proof is complete.

REMARK 1. We remark that the mapping $f: X \to X$ in Theorem 1 of [4] is continuous from the condition of the theorem.

REFERENCES

1. HADZIC, O. Common Fixed Point Theorems for Family of Mappings in Complete Metric Spaces, Math.Japan., 29 (1984), 127-134.
2. JOSHI, M.C. and BOSE, R.K., Some Topics in Nonlinear Functional Analysis, John Wiley and Sons, New York, 1985.
3. ISEKI, K. On Common Fixed Points of Mappings, Bull.Austr.Math.Soc. 10(1975), 365-370.
4. WANG, S.Z., LI, B.Y., GAO, Z.M., and ISEKI, K. Some Fixed Point Theorems on Expansion Mappings, Math.Japan., 29(1984), 631-636.
5. CIRIC, B. A Generalization of Banach's Contraction Principle, Proc. Amer. Math. Soc., 45 (1974), 267-237.
6. GILLESPIE, A.A. and WILLIAMS, B.B. Fixed Point Theorems for Expanding Maps, Applicable Analysis, 14 (1984), 161-165.
7. JUNGCK, G. Commuting Mappings and Fixed Points, Amer.Math.Monthly, 83(1976), 261-263.
8. RHOADES, B.E., Contractive Definitions Revisited, Topological Methods in Nonlinear Functional Analysis, Contemp. Math. Amer. Math. Soc., 21(1983), 190-205.