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A TRIGONOMETRICAL IDENTITY

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1. INTRODUCTION. The object of this note is to establish the following identity:

$$\left\{(1/4)\cot(\theta/2)+\sum_{k=1}^{\infty}\frac{x^{2k}\sin k\theta}{1-x^{2k}}-\frac{1}{2}\sum_{k=1}^{\infty}\frac{x^{k}\sin(k\theta/2)}{1-x^{2k}}\right\}^2$$

$$=\{(1/4)\cot(\theta/2)\}^2-\frac{3}{4}\sum_{k=1}^{\infty}\frac{x^{2k}\cos k\theta}{(1-x^{2k})^2}+\frac{1}{8}\sum_{k=1}^{\infty}\frac{kx^{2k}}{1-x^{2k}}(3-4\cos k\theta)$$

$$+\frac{1}{8}\sum_{k=1}^{\infty}\frac{kx^{k}\cos(k\theta/2)}{1-x^{2k}}-\frac{3}{8}\sum_{k=1}^{\infty}\frac{x^{k}(1+x^{2k})}{(1-x^{2k})^2}\cos(k\theta/2), \tag{1.1}$$

valid for $\theta\in\mathbf{R}$, $x\in\mathbf{C}$, θ not an even multiple of π and $|x|<1$. The details of the proof are supplied in section 2. In our concluding remarks we compare (1.1) with a celebrated identity of Ramanujan, and discuss a uniform method which reveals a total of four such trigonometrical identities.

2. PROOF OF IDENTITY (1.1). Our argument is based on the following variant of the quintuple-product identity:

$$\prod_{1}^{\infty}\frac{(1-x^{2n})(1-a^2x^{2n-2})(1-a^{-2}x^{2n})}{(1+ax^{2n-1})(1+a^{-1}x^{2n-1})}=\sum_{-\infty}^{\infty}x^{n(3n+2)}(a^{-3n}-a^{3n+2}), \tag{1.2}$$

valid for $a,x\in\mathbf{C}$, $a\neq 0$ and $|x|<1$. For a discussion of (1.2) and other forms of the quintuple-product identity see [1].

In (1.2) let $a\to -a$, $x\to x^3$, and multiply the subsequent identity by $-a^{-1}x$ to get

$$(a-a^{-1})x\prod_{1}^{\infty}\frac{(1-x^{6n})(1-a^2x^{6n})(1-a^{-2}x^{6n})}{(1-ax^{6n-3})(1-a^{-1}x^{6n-3})}=\sum_{-\infty}^{\infty}(-1)^n x^{(3n+1)^2}(a^{3n+1}-a^{-3n-1}).$$

Let $F(a, x)$ denote the left side of the foregoing identity, and for a complex variable z, regard zD_z as an operator, where D_z denotes derivation with respect to z. Then,

$$(aD_a)^2 \{F(a, x)\} = \sum_{-\infty}^{\infty} (-1)^n (3n+1)^2 x^{(3n+1)^2} (a^{3n+1} - a^{-3n-1})$$

$$= (xD_x) \{F(a, x)\}. \tag{2.2}$$

We now use the technique of logarithmic differentiation to evaluate the leftmost and rightmost members of (2.2), cancel $F(a, x)$ in the resulting identity, and then let $x \to x^{1/3}$ to get

$$\left\{\frac{a+a^{-1}}{a-a^1} - 2\sum_{k=1}^{\infty} \frac{x^{2k}}{1-x^{2k}}(a^{2k}-a^{-2k}) + \sum_{k=1}^{\infty} \frac{x^k}{1-x^{2k}}(a^k - a^{-k})\right\}^2$$

$$= 1 + \frac{4}{(a-a^{-1})^2} + 4\sum_{k=1}^{\infty} \frac{kx^{2k}}{1-x^{2k}}(a^{2k}+a^{-2k}) - \sum_{k=1}^{\infty} \frac{kx^k}{1-x^{2k}}(a^k+a^{-k})$$

$$-6\sum_{k=1}^{\infty} \frac{kx^{2k}}{1-x^{2k}} + 6\sum_{k=1}^{\infty} \frac{x^{2k}}{(1-x^{2k})^2}(a^{2k}+a^{-2k}) + 3\sum_{k=1}^{\infty} \frac{x^k(1+x^{2k})}{(1-x^{2k})^2}(a^k+a^{-k}).$$

In the foregoing identity let $a = e^{i\theta/2}$, θ subject to the stated restrictions. We simplify the resulting identity, and finally divide by -16 to arrive at identity (1.1).

CONCLUDING REMARKS. The forerunner of all identities of type (1.1) is a celebrated one due to Ramanujan [2, p. 139], viz.,

$$\left\{(1/4)\cot(\theta/2) + \sum_{1}^{\infty} \frac{x^k \sin k\theta}{1-x^k}\right\}^2$$

$$= \{(1/4)\cot(\theta/2)\}^2 + \sum_{1}^{\infty} \frac{x^k \cos k\theta}{(1-x^k)^2} + \frac{1}{2}\sum_{1}^{\infty} \frac{kx^k}{1-x^k}(1-\cos k\theta), \tag{2.4}$$

with the same restrictions on θ and x. Ramanujan himself made substantial applications of his identity to the theory of elliptic modular functions. However, the most familiar application of the identity is perhaps that of Hardy and Wright [3, pp. 311-314]. These authors use the identity to establish Jacobi's formula for the number $r_4(n)$ of representations of a natural number n by sums of four squares. Ewell [4] shows that the method of this note permits an easy and straightforward derivation of Ramanujan's identity. Moreover, the method also reveals two additional trigonometrical identities of this type.

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