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COVERING OF RADIAL SEGMENTS FOR DOMAINS BOUNDED BY k-CIRCLES

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ABSTRACT. Some theorems on radial segments is studied for ring domains bounded by k-circles.

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1. INTRODUCTION.

We designate the chordal distance between the points w_1 and w_2 in the extended complex w-plane $\bar{C}$ by $q(w_1, w_2)$, that is

$$q(w_1,w_2) = |w_1 - w_2| / \sqrt{(1 + |w_1|^2)(1 + |w_2|^2)} \tag{1.1}$$

if w_1 and w_2 are both finite, and

$$q(w_1,\infty) = 1/\sqrt{1 + |w_1|^2}. \tag{1.2}$$

We define the chordal cross ratio of quadruples $w_1, w_2, w_3 w_4$ in $\bar{C}$ by

$$X(w_1,w_2,w_3,w_4) = \frac{q(w_1,w_2)q(w_3,w_4)}{q(w_1,w_3)q(w_2,w_4)}. \tag{1.3}$$

A Jordan curve γ in $\bar{C}$ is called a k-circle, where $0 < k \le 1$, if for all ordered quadruples of points on γ,

$$X(w_1,w_2,w_3,w_4) + X(w_2,w_3,w_4,w_1) \le 1/k. \tag{1.4}$$

This definition of a k-circle was introduced by Blevins [1]. It is well-known that a k-circle is a quasicircle (see [2]). One of the simplest k-circle is $\{w: |\arg w| = \arcsin k\}$. Throughout the note, the value of arcsin is restected between 0 and $\pi/2$.

We consider a class of $C(k)$ of conformal mappings $w = f(z)$ on an annulus $A(R) = \{1 < |z| < R\}$ whose images $D(\gamma) = f(A(R))$ are ring domains γ with inner boundary $f(|z| = 1) = \{|w| = 1\}$ and outer ones k-circles γ. Let w_1 and w_2 be the points on γ such that $w_1 = r_1 e^{i\theta}$ and $w_2 = r_2 e^{i(\theta + \pi)}$ with $0 \le \theta < \pi$. In this note we will consider the problem when the minimum of the values $r_1 r_2$ and $1/r_1 + 1/r_2$ are attained. Corresponding to our problem, recently analogous ones were discussed for the classes of conformal functions by Aharanov and Kirwan [3], and Blevins [4]. They considered the classes of functions conformal in the unit disk but we will do in an annulus, using a simple and elementary method, while their methods are rather complicated. In order to solve our problem the technique of circular symmetrization will be used.

2. LEMMAS.

In this section we summarize the pertinent facts in the following lemmas 2.1-2.5 which are necessary to prove covering theorems in section 3.

LEMMA 2.1[5]. If D is a ring domain and D^* is the circular symmetrization of D with respect to the positive real axis, then

$$Mod\ D \leq Mod\ D^* \tag{2.1}$$

and equality holds if and only if D^* is obtained from D by a simple rotation around the origin.

LEMMA 2.2[6, p. 34]. Let D be a ring domain whose boundary components have spherical diameters $> \delta$ and a mutual spherical distance $< \varepsilon$. Then for $\varepsilon < \delta$, we have

$$Mod\ D \leq \pi^2/\log(\tan\ (\delta/2)/\tan\ (\varepsilon/2)). \tag{2.2}$$

LEMMA 2.3[6, p. 36]. Let $\{B_n\}$ be a sequence of ring domains bounded by a finite number of analytic curves such that $B_n \subset B_{n+1}$ and B_n converges to a ring domain B in the sense of Frechét, then we have

$$\lim_{n \to \infty} Mod\ B_n = Mod\ B. \tag{2.3}$$

LEMMA 2.4[4]. Let D be a ring domain with inner boundary $\{\,|\,w\,|\, = 1\}$ and outer one γ a k-circle. If γ contains the point at infinity and a point w' with $|\,w'\,| = a$, then the circular symmetrization D^* of D with respect to the positive real axis is contained in the domain

$$D(k,a) = \{w\colon |\arg\,(w+a)\,| < \pi - \arcsin\,k\} \cap \{\,|\,w\,|\, > 1\}. \tag{2.4}$$

Now we prove the following lemma which plays important roles in section 3.

LEMMA 2.5. Let $w = f(z)$ be a function $C(k)$ and $\gamma = f(\,|\,z\,|\, = R)$ contain the point at infinity. Then for the distance $d(\gamma, 0)$ between the origin and γ, there holds the inequality

$$d(\gamma, 0) \geq a_0, \tag{2.5}$$

where a_0 is a positive constant uniquely determined from the relation $Mod\ D(k, a_0) = \log\ R$ for fixed values R and k. The equality (2.5) holds if and only if $D(\gamma)$ is $D(k, a_0)$ except for a simple rotation around the origin.

PROOF. At first we will verify that the equation

$$Mod\ D(k,a) = \log\ R \tag{2.6}$$

has a unique solution $a = a_0$. $Mod\ D(k,a)$ is a strictly increasing function of a variable a. Since $\lim_{a \to \infty} Mod\ D(k,a) = \infty$ and from Lemma 2.2 $\lim_{a \to \infty} Mod\ D(k,a) = 0$, there exist a_1 and a_2 such that $a_1 < a_2$ and

$$Mod\ D(k, a_1) < \log R < Mod\ D(k, a_2). \tag{2.7}$$

Continuity of the module holds for the sequence of ring domains bounded by a finite number of analytic curves from Lemma 2.3. Therefore the equation (2.6) has a unique solution $a = a_0$ for fixed values k and R.

Let w' be a point on γ such that $|\,w'\,| = d(\gamma, 0)\ (= a)$. We consider the circular symmetrization $D^*(\gamma)$ of $D(\gamma)$ with respect to the positive real axis. Using Lemma 2.1, 2.4 and monotonicity of the module, we have the inequalities

$$Mod\ D(\gamma) \leq Mod\ D^*(\gamma) \leq Mod\ D(k,a) \quad \text{with} \quad Mod\ D(k, a_0) = \log\ R \tag{2.8ab}$$

where equality

$$Mod\ D(\gamma) = Mod\ D(k,a) \tag{2.9}$$

holds if and only if $D(\gamma)$ is obtained from $D(k,a)$ by a simple rotation around the origin. From the relation

$$Mod\ D(\gamma) = Mod\ D(k, a_0)\ (= \log\ R), \tag{2.10}$$

$$Mod\ D(\gamma) \leq Mod\ D(k,a) \tag{2.11}$$

and monotonicity of the module, we have

$$a \geq a_0 \tag{2.12}$$

which implies the desired inequality (2.5). Using Lemma 2.1 we conclude that equality in (2.5) holds if and only if $D(\gamma)$ is $D(k, a_0)$ except for a simple rotation around the origin.

3. COVERING OF RADIAL SEGMENTS.

In this section we will prove the following covering theorems on radial segments.

THEOREM 3.1. Let $w = f(z)$ be a function in $C(k)$, and w_1 and w_2 be the points on $\gamma = f(|z| = R)$ such that $w_1 = r_1 e^{i\theta}$, $w_2 = r_2 e^{i(\theta+\pi)}$ with $0 \leq \theta < \pi$. Then we have

$$r_1 r_2 \geq (a_0 + \sqrt{a_0^2 - 1})^2. \tag{3.1}$$

The equality holds if and only if $f(A(R))$ is the image of $D(k, a_0)$ under the mapping

$$\zeta(w) = (1 + (a_0 - \sqrt{a_0^2-1})w)/(w + a_0 - \sqrt{a_0^2 - 1}) \tag{3.2}$$

except a simple rotation around the origin.

PROOF. At first we estimate $\min_{0 \leq \theta < \pi}(r_1 + r_2)$ and then we apply the result to proving the inequality (3.1). Without of generality we can assume (considering a rotation if necessary)

$$\min_{0 \leq \theta < \pi}(r_1 + r_2) = b + bt \quad (t \geq 1,\ b > 1) \tag{3.3}$$

and $b,\ -bt \in \gamma$. We consider a Möbius transformation

$$\zeta(w) = (1 + tbw)/(w + tb) \tag{3.4}$$

which maps $f(A(R))$ onto a ring domain $D(\gamma')$ whose boundary consists of inner boundary $\{|\zeta| = 1\}$ and outer one γ'. Since the chordal cross ratio is invariant under Möbius transformations, γ' is a k-circle. By the mapping (3.4), b and $-bt$ are transformed onto $(1 + tb^2)/(b + tb)$ and the point at infinity, respectively.

Using Lemma 2.5 we have

$$(1 + tb^2)/(b + bt) \geq a_0 \tag{3.5}$$

which implies

$$b \geq a_0(1+t)/2t + (1/2t)\sqrt{a_0^2(1+t)^2 - 4t} \quad \text{or} \quad b \leq a_0(t+1)/2t - (1/2t)\sqrt{a_0^2(1+t)^2 - 4t}. \tag{3.6ab}$$

When γ contains the point at infinity, the inequality (3.6b) never holds, because $a_0 \leq b$ follows from Lemma 2.5 and $a_0(1+t)/2t - (1/2t)\sqrt{a_0^2(1+t)^2 - 4t} < a_0$. Thus we have

$$\begin{aligned} b + bt = (1+t)b &\geq a_0(1+t)^2/2t + ((1+t)/2t)\sqrt{a_0^2(1+t)^2 - 4t} \\ &= a_0(1+t)^2/2t + ((1+t)/\sqrt{t})\sqrt{a_0^2(1+t)^2/4t - 1} = 2(a_0 + \sqrt{a_0^2 - 1}). \end{aligned} \tag{3.7}$$

Now we consider the case when γ does not contain the point at infinity. Without loss of generality we can assume $a = d(\gamma, 0) \in \gamma$. For a point $-d$ (<0) on γ, a Möbius transformation $\zeta(w) = (1 + dw)/(w + d)$ maps the points a and $-d$ onto $(1 + ad)/(a + d)(<a)$ and the point at infinity, respectively. This means that the minimum of $d(\gamma, 0)$ in $C(k)$ is attained (if and) only if γ contains the point at infinity. Therefore the inequality (3.7) never holds even when γ does not contain the point at infinity. Now we consider another Möbius transformation

$$\zeta(w) = (w_1/\bar{w}_1)\,(1 - \bar{w}_1 w)/(w - w_1). \tag{3.8}$$

By this transformation (3.8) $D(\gamma)$ is mapped onto a domain $D(\gamma')$ whose boundary consists of inner boundary $\{|\zeta| = 1\}$ and outer γ' a k-circle. Using Lemma 2.5 we have

$$|\zeta(w_2)| \geq a_0 \tag{3.9}$$

which implies, substituting $w_1 = r_1 e^{i\theta}$ and $w_2 = r_2 e^{i(\theta + \pi)}$,

$$(1 + r_1 r_2)/(r_1 + r_2) \geq a_0, \tag{3.10}$$

and then

$$r_1 r_2 \geq a_0 (r_1 + r_2) - 1. \tag{3.11}$$

Combining the relation (3.7) and (3.11) we have

$$r_1 r_2 \geq 2a_0(a_0 + \sqrt{a_0^2 - 1}) - 1 = (a_0 + \sqrt{a_0^2 - 1})^2. \tag{3.12}$$

If $f(A(R))$ is the image of $D(k, a_0)$ under the mapping (3.2), the equality in (3.1) holds for $w_1 = a_0 + \sqrt{a_0^2 - 1}$ and $w_2 = -a_0 - \sqrt{a_0^2 - 1}$. On the other hand, considering the case when the equalities hold, we can easily conclude that the equality in (3.1) holds only if $f(A(R))$ is the image of $D(k, a_0)$ under the mapping (3.2) except a simple rotation around the origin.

As the application of Theorem 3.1 we have the following theorem.

THEOREM 3.2. Under the assumption as described in Theorem 3.1, we have

$$1/r_1 + 1/r_2 \leq 2/(a_0 + \sqrt{a_0^2 - 1}). \tag{3.13}$$

The equality holds if and only if $F(A(R))$ is the image of $D(k, a_0)$ under the mapping (3.2) except a simple rotation around the origin.

PROOF. We use the inequality (3.10) obtained in proof of Theorem 1 we have

$$r_1 r_2/(r_1 + r_2) \geq a_0 - 1/(r_1 + r_2), \tag{3.14}$$

and then using the inequality (3.7) we have

$$(1/r_1 + 1/r_2)^{-1} \geq a_0 - 1/(2(a_0 + \sqrt{a_0^2 - 1})) = (a_0 + \sqrt{a_0^2 - 1})/2. \tag{3.15}$$

This implies the desired inequality (3.13). We omit to estimate the case when the equality in (3.13) holds, because the proof is similar to one in Theorem 3.1.

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