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OSCILLATION CRITERIA IN NEUTRAL EQUATIONS OF n ORDER WITH VARIABLE COEFFICIENTS

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ABSTRACT. Consider the n-order neutral delay differential equation

$$\frac{d^n}{dt^n}\left[y(t)+P(t)y(t-\tau)\right]+Q(t)y(t-\sigma)=0$$

where $P, Q \in C[[t_0,\infty),\mathbb{R}]$ and the delays τ and σ are nonnegative real numbers. In this paper we examined the oscillatory behavior of the solutions of the above equation using techniques which allow the relaxation of the restrictions which has been introduced previously. We illustrate new type of conditions which improve and extend known results, by relaxing hypotheses that P is constant and Q is τ-periodic.

KEY WORDS AND PHRASES. Neutral delay differential equations, τ-periodic coefficient, oscillatory behavior.
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1. INTRODUCTION AND PRELIMINARIES. A neutral delay differential equation (NDDE for short) is a differential equation in which the higher order derivative of the unknown function appears in the equation both with and without delays.

Consider the n^{th} order NDDE with variable coefficients

$$\frac{d^n}{dt^n}[y(t)+P(t)y(t-\tau)]+Q(t)y(t-\sigma)=0 \tag{1.1}$$

where

$$P \in C[[t_0,\infty),\mathbb{R}], Q \in C[[t_0,\infty),\mathbb{R}^+] \text{ and } \tau,\sigma \in \mathbb{R}^+. \tag{1.2}$$

When $n=1$, the oscillatory behavior of Eq. (1) has been investigated in [1], [2], [3] and [4]. While for general cases, some oscillatory results for Eq. (1) have also been obtained in [5] and [6].

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In this paper we also study the oscillatory behavior of the solutions of Eq. (1.1). In Section 2 we establish sufficient conditions for the oscillation of all solutions of Eq. (1) with $n = 1$, by relaxing the restrictions which have been introduced in [1], [2], [3] and [4]. In Section 3 we obtain several sufficient conditions for the oscillation of all solutions of Eq. (1) with n odd. This new type of conditions improve and extend results which have been established in [6], by relaxing the hypotheses that P is constant and Q is τ-periodic.

Let $\rho = \max\{\tau,\sigma\}$. By a solution of Eq. (1) we mean a function $y \in C[t_1 - \rho, \infty), \mathbf{R}]$, for some $t_1 \geq t_0$, such that $y(t) + P(t)y(t-\tau)$ is n times continuously differentiable on $[t_1, \infty)$ and such that Eq. (1) is satisfied for $t \geq t_1$.

As usual, a solution of Eq. (1.1) is called oscillatory if it has arbitrarily large zeros and nonoscillatory if it is eventually positive or eventually negative.

In the sequel, for convenience, when we write an inequality without specifying its domain of validity we will assume that it holds for all large t.

2. SUFFICIENT CONDITIONS FOR THE OSCILLATION OF THE FIRST ORDER NDDE.

Throughout this section, and without any further reference to, we consider Eq. (1.1) with $n = 1$. We will establish sufficient conditions for the oscillation of all solutions of the first order NDDE (1), which differ from the corresponding conditions in [1], [3], [4] and [6] in terms of τ and σ.

The following lemma extracted from [1] will be utilized in the proofs of the main results in this section.

LEMMA 1. Assume that (1.2) holds and

$$\int_{t_o}^{\infty} Q(t)ds = \infty. \tag{2.1}$$

Let $y(t)$ be an eventually positive solution of Eq. (1.1) and set

$$z(t) = y(t) + P(t)y(t-\tau). \tag{2.2}$$

Then the following statements are true:

(i) If $-1 \leq P(t) \leq 0$ then eventually $z(t) > 0$ and $z'(t) \leq 0$ (2.3)

(ii) If $P(t) \leq -1$ then eventually $z(t) < 0$ and $z'(t) \leq 0$. (2.4)

Now, for the sake of convenience, we set

$$R(t) = P(t-\sigma)\frac{Q(t)}{Q(t-\tau)} \text{ for } Q(t) > 0. \tag{2.5}$$

THEOREM 1. Assume that (1.2) and (2.1) hold,

$$-1 \leq P(t) \leq 0,\ Q(t) > 0,\ R'(t) \leq 0 \tag{2.6}$$

and

$$1 + R(t)\int_{t-\tau}^{t+\sigma} Q(s)ds \leq 0. \tag{2.7}$$

Then every solution of Eq. (1.1) oscillates.

PROOF. Assume, for the sake of contradiction, that Eq. (1.1) has an eventually positive solution $y(t)$. Define $z(t)$ by (4). Then by Lemma 1, (2.3) holds. Also, $z(t)$ satisfies the equation

$$z'(t) + R(t)z'(t-\tau) + Q(t)z(t-\sigma) = 0. \tag{2.8}$$

By integrating both sides of (2.8) from t to $t+\tau+\sigma$ and by noting (2.3), (2.6) and the fact that

$$\int_t^{t+\tau+\sigma} R(s)z'(s-\tau)ds \geq R(t)\int_t^{t+\tau+\sigma} z'(s-\tau)ds = R(t)[z(t+\sigma) - z(t-\tau)]$$

we are led to the inequality

$$[z(t+\tau+\sigma)-z(t)]+R(t)[z(t+\sigma)-z(t-\tau)]+z(t+\tau)\int_t^{t+\tau+\sigma}Q(s)ds\leq 0. \tag{2.9}$$

As $R(t)\leq 0$ and $z'(t)\leq 0$ it follows from (2.9) that

$$z(t+\tau+\sigma)+z(t+\tau)\int_t^{t+\tau+\sigma}Q(s)ds\leq z(t)$$

or

$$z(t+\sigma)+z(t)\int_{t-\tau}^{t+\sigma}Q(s)ds\leq z(t-\tau). \tag{2.10}$$

On the other hand we rewrite (2.9) as

$$[z(t+\tau+\sigma)+z(t+\tau)\int_t^{t+\tau+\sigma}Q(s)ds]-z(t)+R(t)z(t+\sigma)-R(t)z(t-\tau)\leq 0. \tag{2.11}$$

Then by combining (2.10) and (2.11) we find

$$[z(t+\tau+\sigma)+z(t+\tau)\int_t^{t+\tau+\sigma}Q(s)ds]-z(t)-R(t)z(t)\int_{t-\tau}^{t+\sigma}Q(s)ds\leq 0$$

or

$$[z(t+\tau+\sigma)+z(t+\tau)\int_t^{t+\tau+\sigma}Q(s)ds]-z(t)[1+R(t)\int_{t-\tau}^{t+\sigma}Q(s)ds]\leq 0.$$

This is a contradiction and the proof of the theorem is complete.

By noting the fact that if $Q(t)$ is τ-periodic then $z(t)$ satisfies the equation

$$z'(t)+P(t-\sigma)z'(t-\tau)+Q(t)z(t-\sigma)=0 \tag{2.12}$$

and by an argument similar to that in the proof of the Theorem 1 one can see that the following result holds.

THEOREM 2. Assume that (2.1) and (2.1) hold,

$$-1\leq P(t)\leq 0,\ P'(t)\leq 0,\ Q(t) \text{ is } \tau\text{-periodic}$$

and

$$1+P(t-\sigma)\int_{t-\tau}^{t+\sigma}Q(s)ds\leq 0.$$

Then every solution of Eq. (1.1) oscillates.

REMARK 1. For $-1\leq P(t)\leq 0$, all the oscillatory results for Eq. (1.1) in [1], [2], [3] and [4] always assume that either $\sigma\neq 0$ or $\sigma>\tau$. It is easy to see that there are no such restrictions in Theorems 1 and 2.

EXAMPLE 1. The neutral delay differential equation

$$\frac{d}{dt}[y(t)+(\frac{1}{t-1}-1)y(t-2)]+(1-\frac{1}{t+1})y(t)=0$$

satisfies the hypotheses of Theorem 1. Therefore every solution of this equation oscillates.

THEOREM 3. Assume that (1.2) and (2.1) hold,

$$P(t)\leq -1,\ Q(t)>0,\ R'(t)\leq 0 \tag{2.13}$$

and

$$1-\frac{1}{R(t-\sigma)}\int_{t-2\tau}^{t-\sigma}\frac{Q(s+\tau)}{R(s+\tau)}ds\leq 0. \tag{2.14}$$

Then every solution of Eq. (1.1) oscillates.

PROOF. Otherwise, Eq. (1.1) has an eventually positive solution $y(t)$. Define $z(t)$ by (4). Then by Lemma 1 (ii), (2.4) holds. Also $z(t)$ satisfies (2.8) and so

$$z'(t-\tau)+\frac{1}{R(t)}z'(t)+\frac{Q(t)}{R(t)}z(t-\sigma)=0. \tag{2.15}$$

From (2.14) we see that $2\tau > \sigma$. Thus by integrating both sides of (2.15) from $t-2\tau$ to $t-\sigma$ and by noting (2.4), (2.13) and the fact that

$$\int_{t-2\tau}^{t-\sigma}\frac{1}{R(s)}z'(s)ds \geq \frac{1}{R(t-\sigma)}\int_{t-2\tau}^{t-\sigma}z'(s)ds = \frac{1}{R(t-\sigma)}[z(t-\sigma)-z(t-2\tau)]$$

we obtain

$$[z(t-\tau-\sigma)-z(t-3\tau)]+\frac{1}{R(t-\sigma)}[z(t-\sigma)-z(t-2\tau)]+z(t-2\tau-\sigma)\int_{t-2\tau}^{t-\sigma}\frac{Q(s)}{R(s)}ds \leq 0. \tag{2.16}$$

This inequality, in view of $R(t)<0$ and $z'(t)\leq 0$, implies that

$$z(t-\tau-\sigma)-z(t-3\tau)+z(t-2\tau-\sigma)\int_{t-2\tau}^{t-\sigma}\frac{Q(s)}{R(s)}ds \leq 0$$

or

$$z(t-\sigma)-z(t-2\tau) \leq -z(t-\tau-\sigma)\int_{t-2\tau}^{t-\sigma}\frac{Q(s+\tau)}{R(s+\tau)}ds \leq 0. \tag{2.17}$$

On the other hand from (2.16) we also have

$$[-z(t-3\tau)+z(t-2\tau-\sigma)\int_{t-2\tau}^{t-\sigma}\frac{Q(s)}{R(s)}ds]+z(t-\tau-\sigma)+\frac{1}{R(t-\sigma)}[z(t-\sigma)-z(t-2\tau)] \leq 0. \tag{2.18}$$

Then by combining (2.17) and (2.18) we are led to the inequality

$$[-z(t-3\tau)+z(t-2\tau-\sigma\int_{t-2\tau}^{t-\sigma}\frac{Q(s)}{R(s)}ds]+z(t-\tau-\sigma)-\frac{1}{R(t-\sigma)}z(t-\tau-\sigma)\int_{t-2\tau}^{t-\sigma}\frac{Q(s+\tau)}{R(s+\tau)}ds \leq 0$$

or

$$[-z(t-3\tau)+z(t-2\tau-\sigma)\int_{t-2\tau}^{t-\sigma}\frac{Q(s)}{R(s)}ds]+z(t-\tau-\sigma)[1-\frac{1}{R(t-\sigma)}\int_{t-2\tau}^{t-\sigma}\frac{Q(s+\tau)}{R(s+\tau)}ds] \leq 0.$$

This is a contradiction and the proof is complete.

By noting the fact that $z(t)$ satisfies (2.11) for $Q(t)$ τ-periodic and by an argument similar to that in the proof of Theorem 3 we can obtain the following result.

THEOREM 4. Assume that (1.2) and (2.1) hold,

$$P(t) \leq -1,\ P'(t) \leq 0,\ Q(t) \text{ is } \tau\text{-periodic}$$

and

$$1-\frac{1}{P(t-2\sigma)}\int_{t-2\tau}^{t-\sigma}\frac{Q(s+\tau)}{P(s+\tau-\sigma)}ds \leq 0.$$

Then every solution of Eq. (1.1) oscillates.

REMARK 2. All the oscillatory results for Eq. (1.1) with $P(t) \leq -1$ in [1], [2], [3] and [4] assume that $\tau > \sigma$. Here, we reduce this restriction to $2\tau > \sigma$ in Theorems 3 and 4.

EXAMPLE 2. The neutral delay differential equation

$$\frac{d}{dt}[y(t)+(\tfrac{1}{t}-2)y(t-1)]+4(1+\sin 2\pi t)y(t-1)=0,\ t \geq 1$$

satisfies the hypotheses of Theorem 4. Therefore every solution of this equation oscillates.

3. SUFFICIENT CONDITIONS FOR THE OSCILLATION OF ODD ORDER NDDE.

In this section we will establish some sufficient conditions for the oscillation of all solutions of Eq. (1.1) with n odd.

Throughout this section we will assume that n is odd and $Q(t)$ is positive in Eq. (1.1) without specifying them.

First we present a lemma which is extracted from [5] and [6] and will be utilized in the proofs of the main results in this section.

LEMMA 2. Assume that (1.2) and (2.1) hold and

$$P(t) \text{ is bounded.} \tag{3.1}$$

Let $y(t)$ be an eventually positive solution of Eq. (1.1) and set

$$z(t) = y(t) + P(t)y(t-\tau). \tag{3.2}$$

Then the following statements are true:

(i) Assume that

$$P(t) \leq -1,\ P(t) \not\equiv -1 \text{ on } [T,\infty) \text{ for any } T \geq t_o. \tag{3.3}$$

Then for each $i = 0,1,...,n-1$

$$\lim_{t\to\infty} z^{(i)}(t) = -\infty. \tag{3.4}$$

(ii) Assume that

$$\text{either } -1 \leq P(t) \leq 0 \text{ or } 0 \leq P(t) \leq 1. \tag{3.5}$$

Then

$$\lim_{t\to\infty} z'^{(i)}(t) = 0 \text{ for each } i = 0,1,...,n-1$$

and (3.6ab)

$$z(t) > 0,\ z(t) < 0,..., z^{(n-1)}(t) > 0,\ z^{(n)} < 0.$$

(iii) Assume that

$$P(t) \geq -1 \tag{3.7}$$

and there exists a positive number q such that

$$Q(t) \geq q. \tag{3.8}$$

Then (3.6ab) holds.

Let $y(t)$ be a solution of Eq. (1.1). Define $z(t)$ by (3.2) and

$$w(t) = z(t) + R(t)z(t-\tau) \tag{3.9}$$

where

$$R(t) = P(t-\sigma)\frac{Q(t)}{Q(t-\tau)}.$$

By direct substitution one can see that $z(t)$ is an n times continuously differentiable solution of the NDDE

$$z^{(n)}(t) + R(t)z^{(n)}(t-\tau) + Q(t)z(t-\sigma) = 0 \tag{3.10}$$

THEOREM 5. Assume that (1.2), (3.1) and (3.3) hold,

$$R \in C^n[[t_0,\infty),\mathbb{R}],\ R(t) < -1 \text{ and } R^{(i)}(t) \leq 0 \text{ for } i = 1,2,...n. \tag{3.11}$$

Suppose also there exists a positive number r such that

$$\frac{Q(t)}{1+R(t+\tau-\sigma)} \le -r \text{ and } r^{1/n}(\tau-\sigma)/n > \frac{1}{e}. \tag{3.12ab}$$

Then every solution of Eq. (1.1) oscillates.

PROOF. Assume, for the sake of contradiction, that Eq. (1.1) has an eventually positive solution $y(t)$. Define $z(t)$ and $w(t)$ by (3.2) and (3.9), respectively. From (3.11) and (3.12ab) it is easy to see that $Q(t)$ is bounded from below by a positive number, say q. Thus (2.1) holds and by Lemma 2 (i), (3.4) holds. Hence for t sufficiently large,

$$\begin{aligned} w^{(n)}(t) &= z^{(n)}(t) + [R(t)z(t-\tau)]^{(n)} \\ &= z^{(n)}(t) + \sum_{i=0}^{n} C_n^i R^{(i)}(t) z^{(n-i)}(t-\tau) \\ &= -Q(t)z(t-\sigma) + \sum_{i=1}^{n} C_n^i R^{(i)}(t) z^{(n-i)}(t-\tau) \\ &\ge -qz(t-\sigma) + \sum_{i=1}^{n} C_n^i R^{(i)}(t) z^{(n-i)}(t-\tau) \end{aligned}$$

which, in view of (3.4) and (3.11), implies that

$$\lim_{t\to\infty} w^{(i)}(t) = \infty \text{ for } i = 0, 1, \dots, n.$$

Hence eventually

$$w(t) > 0. \tag{3.13}$$

Next observe that

$$w(t) = z(t) + R(t)z(t-\tau) \le [1+R(t)]z(t-\tau)$$

and so

$$\begin{aligned} -\frac{Q(t)}{1+R(t+\tau-\sigma)} w(t+\tau-\sigma) &\le -Q(t)z(t-\sigma) \\ &= w^{(n)}(t) - \sum_{i=1}^{n} C_n^i R^{(i)}(t) z^{(n-i)}(t-\tau) \end{aligned}$$

or

$$w^{(n)}(t) - \frac{Q(t)}{-[1+R(t+\tau-\sigma)]} w(t+(\tau-\sigma)) \ge \sum_{i=1}^{n} C_n^i R^{(i)}(t) z^{(n-i)}(t-\tau) \ge 0. \tag{3.14}$$

It is known however, see [7], that under the hypotheses (3.12a) and (3.12b) the inequality (3.14) cannot have an eventually positive solution. This contradicts (3.13) and completes the proof of the theorem.

The following result is an immediate corollary of Theorem 5 which has been established in [6].

COROLLARY 1. Assume that (1.2) and (3.8) hold,

$$P(t) \equiv p < -1 \text{ and } Q(t) \text{ is } \tau\text{-periodic.}$$

Suppose also that

$$\left[-\frac{q}{1+p}\right]^{\frac{1}{n}} \frac{\tau-\sigma}{n} > \frac{1}{e}.$$

Then every solution of Eq. (1.1) oscillates.

EXAMPLE 3. The neutral delay differential equation

$$\frac{d^3}{dt^3}[y(t)-(1+\frac{1}{t+1})y(t-2)]+e^{t^2}y(t-1)=0$$

satisfies the hypotheses of Theorem 5. Hence every solution of this equation oscillates.

Theorem 6. Assume that (1.2), (2.1) and (3.1) hold,

$$R\in C^n[[t_0,\infty),\mathbb{R}],\ R(t)>-1 \text{ and } (-1)^iR^{(i)}(t)\geq 0 \text{ for } i=1,2,...n \tag{3.15}$$

Suppose also that there exists a positive number r such that

$$\frac{Q(t)}{1+R(t+\tau-\sigma)}\geq r \text{ and } r^{1/n}(\sigma-\tau)/n>\tfrac{1}{e}. \tag{3.16ab}$$

Then every solution of Eq. (1.1) oscillates in each of the following cases.
(i) (3.5) holds; (ii) (3.7) and (3.8) hold.

PROOF. Assume that one of the conditions is satisfied and that, contrary to the conclusion of the theorem, Eq. (1.1) has an eventually positive solution $y(t)$. Consider $z(t)$ and $w(t)$ as they have been defined by (3.2) and (3.9), respectively. Then by Lemma 2 (ii) and (iii), (3.6ab) holds.

From (3.15) we see that for each $i=0,1,...,n-1, R^{(i)}(t)$ is bounded. Hence, by nothing (3.6ab) and the fact that

$$w^{(i)}(t)=z^{(i)}(t)+[R(t)z(t-\tau)]^{(i)}$$

$$=z^{(i)}(t)+\sum_{j=0}^{i}C_i^jR^{(j)}(t)z^{(i-j)}(t-\tau)$$

we find

$$\lim_{t\to\infty}w^{(i)}(t)=0 \text{ for } i=0,1...,n-1. \tag{3.17}$$

Also from (3.10), (3.15) and (3.6ab) it follows that

$$w^{(n)}(t)=-Q(t)z(t-\sigma)+\sum_{i=1}^{n}C_n^iR^{(i)}(t)z^{(n-i)}(t-\tau)<0. \tag{3.18}$$

Hence (3.17) and (3.19) imply that

$$w(t)>0,\ w'(t)<0,...,w^{(n-1)}(t)>0,w^{(n)}(t)<0. \tag{3.19}$$

Next observe that

$$w(t)=z(t)+R(t)z(t-\tau)\leq[1+R(t)]z(t-\tau)$$

we find

$$\frac{Q(t)}{1+R(t+\tau-\sigma)}w(t+\tau-\sigma)\leq Q(t)z(t-\sigma)=-w^{(n)}(t)+\sum_{i=1}^{n}C_n^iR^{(i)}(t)z^{(n-i}(t-\tau)$$

or

$$w^{(n)}(t)+\frac{Q(t)}{1+R(t+\tau-\sigma)}w(t-(\sigma-\tau))\leq\sum_{i=1}^{n}C_n^iR^{(i)}(t)z^{(n-i)}(t-\tau)\leq 0. \tag{3.20}$$

It is known however that under the hypotheses (3.16a) and (3.16b) the inequality (3.20) cannot have an eventually positive solution, see [8] for details. This contradicts (3.19) and completes the proof of the theorem.

The following result is an immediate corollary of Theorem 6.

COROLLARY 2. Assume that (1.2) holds, $Q(t)$ is τ-periodic and that

$$P \in C^n[[t_0,\infty),\mathbf{R}],\ P(t) > -1 \text{ and } (-1)^i P^{(i)}(t) \geq 0 \text{ for } i = 1,2,\ldots,n.$$

Suppose that there exists a positive number r such that

$$\frac{Q(t)}{1+P(t+\tau-\sigma)} \geq r \text{ and } r^{1/n}(\sigma-\tau)/n > \frac{1}{e}.$$

Then every solution of Eq. (1.1) oscillates.

REMARK 3. In the specific case $P(t) \equiv p > -1$ the conclusion of Corollary 2 appears in [6].

EXAMPLE 4. The neutral delay differential equations

$$\frac{d^5}{dt^5}[y(t) + (\frac{1}{t+1} - 1)y(t-1)] + (\tfrac{3}{t})y(t-2) = 0,\ t \geq 1 \tag{3.21}$$

and

$$\frac{d^3}{dt^3}[y(t) + (1-\tfrac{1}{t})y(t-2)] + ty(t-3) = 0,\ t \geq 1 \tag{3.22}$$

satisfy the hypotheses of Theorem 6(i). Therefore, every solution of (3.21) and (3.22) oscillates.

EXAMPLE 5. The neutral delay differential equation

$$\frac{d^3}{dt^3}[y(t) + (1+e^{-t})y(t-\pi)] + (2+\sin^2 t)y(t-2\pi) = 0,\ t \geq 0 \tag{3.23}$$

satisfies the hypotheses of Theorem 6(ii). Hence, every solution of Eq. (3.23) oscillates.

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