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FINITE-INFINITE-RANGE INEQUALITIES IN THE COMPLEX PLANE

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ABSTRACT. Let $E \subseteq \mathbf{C}$ be closed, w be a suitable weight function on E, σ be a positive Borel measure on E. We discuss the conditions on w and σ which ensure the existence of a fixed compact subset K of E with the following property. For any p, $0 < p \leq \infty$, there exist positive constants c_1, c_2 depending only on E, w, σ and p such that for every integer $n \geq 1$ and every polynomial P of degree at most n,

$$\int_{E\setminus K} |w^n P|^p d\sigma \leq c_1 \exp(-c_2 n) \int_K |w^n P|^p d\sigma.$$

In particular, we shall show that the support of a certain extremal measure is, in some sense, the smallest set K which works. The conditions on σ are formulated in terms of certain localized Christoffel functions related to σ.

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1. INTRODUCTION.

Let $\alpha > 1$ and $w_\alpha(x) := \exp(-|x|^\alpha)$, $x \in \mathbf{R}$. A special case of an inequality proved by G. Freud [1] is the following. There exist positive constants c_1, c_2, c_3 depending only on α such that for every integer $n \geq 1$ and polynomial P of degree at most n,

$$\int_{|x|\geq c_1} |w_\alpha^n(x)P(x)|^2 dx \leq c_2 \exp(-c_3 n) \int_{|x|\leq c_1} |w_\alpha^n(x)P(x)|^2 dx. \qquad (1.1)$$

An inequality of the form (1.1), known as a *finite-infinite range inequality,* is of critical importance in the study of weighted polynomial approximation on $\mathbf{R}$. In the past few years, many mathematicians have investigated inequalities of this form in great detail (cf. [3] for example). In particular, Mhaskar and Saff [4] have obtained a highly generalized and precise form of (1.1), using potential theoretic ideas. These ideas, in turn, have been extended to the case when the underlying sets are subsets of the complex plane $\mathbf{C}$ rather than the real line. In studying the finite-infinite range inequalities on subsets of $\mathbf{C}$, an immediate problem which must be solved is to decide upon a natural choice of a measure to replace the one-dimensional Lebesgue measure which is so natural for subsets of the real line. In particular, one has to include the arc-length measure on sufficiently smooth curves as well as the area measure on domains and their combinations.

In this paper, we define one such class of 'natural' measures and then extend the results of [4] to the complex domain. Our results will demonstrate the importance of Christoffel functions in this theory. In turn, we estimate these Christoffel functions when the measure under consideration satisfies certain density conditions. We also describe certain applications to the theory of extremal polynomials in the complex plane.

In Section 2, we develop the necessary technical background including some of the definitions. The main results and applications will be described in Section 3. The proofs of the new results will be given in Section 4.

2. PREPARATORY INFORMATION.

We say that a function $w : \mathbf{C} \to [0, \infty)$ is *admissible* (or an admissible weight function) if each of the following conditions (W1), (W2), (W3) holds.

(W1) w is upper semi-continuous.

(W2) The set $\Sigma_0(w) := \{z : w(z) > 0\}$ has nonzero capacity.

(W3) If $\Sigma_0(w)$ is unbounded, then $|zw(z)| \to 0$ as $|z| \to \infty$, $z \in \Sigma_0(w)$.

Here and throughout this paper, the term 'capacity' means the inner logarithmic capacity (cf. [13], p.55). For any set $A \subseteq \mathbf{C}$, its capacity will be denoted by cap(A). A property is said to hold quasi-everywhere (q.e.) on a set A if the subset of A where it does not hold is of capacity zero. Let E be a closed subset of $\mathbf{C}$. A function w is called *admissible on E* if the function w_E that coincides with w on E and is zero on $\mathbf{C} \backslash E$ is admissible. Typical examples of admissible weight functions are (i)$\exp(-|z|^\alpha)$, $\alpha > 0$, which are admissible on every closed subset of $\mathbf{C}$ having positive capacity and (ii) any positive continuous function on any compact subset of $\mathbf{C}$ which has positive capacity. Let $\mathcal{M}(E)$ denote the class of all unit positive Borel measures supported on E. If $\sigma \in \mathcal{M}(E)$, then its potential is defined by

$$U(\sigma, z) := \int_E \log(1/|z-t|)d\sigma(t), \qquad z \in \mathbf{C}. \tag{2.1}$$

The (w-modified) energy of σ is defined by

$$I(w, \sigma) := \int_E U(\sigma, z)d\sigma(z). \tag{2.2}$$

We define

$$V(w, E) := \inf\{I(w, \sigma) : \sigma \in \mathcal{M}(E)\}, \tag{2.3}$$

$$Q(z) := \log(1/w(z)), \qquad z \in \mathbf{C}. \tag{2.4}$$

For integer $n \geq 0$, Π_n will denote the class of all algebraic polynomials of degree at most n. Finally, for any set $A \subseteq \mathbf{C}$ and a complex valued function g on A, we write

$$||g||_A := \sup_{z\in A} |g(z)|. \tag{2.5}$$

Our starting point is the following theorem proved in [5] in the case when $E \subseteq \mathbf{R}$. The version stated below can be proved in exactly the same way as in [5] with obvious and minor modifications.

THEOREM 2.1. *Let $E \subseteq \mathbf{C}$ be closed, w be an admissible function on E.*

(a) The quantity $V(w, E)$ defined in (2.3) is finite.

(b) There exists a unique element $\mu := \mu(w, E) \in \mathcal{M}(E)$ such that $I(w, \mu) = V(w, E)$. This measure μ has finite logarithmic energy.

(c) The set $\mathcal{S} := \mathcal{S}(w, E) := supp(\mu)$ is a compact subset of $\Sigma_0(w) \cap E$, and $\mathrm{cap}(\mathcal{S}) > 0$. Moreover, Q is bounded from above on $\mathcal{S}$.

(d) With

$$F := F(w, E) := V(w, E) - \int Q d\mu, \tag{2.6}$$

we have

$$U(\mu, z) + Q(z) \geq F, \qquad q.e. \text{ on } E. \tag{2.7}$$

(e) We have

$$U(\mu, z) + Q(z) \leq F, \qquad \text{for every } z \in \mathcal{S}. \tag{2.8}$$

In particular, the equation

$$U(\mu, z) + Q(z) = F \tag{2.9}$$

holds q.e. on $\mathcal{S}$.

(f) For any positive integer n and $P \in \Pi_n$, if

$$|w^n(z)P(z)| \leq M \qquad q.e. \text{ on } \mathcal{S}, \tag{2.10}$$

then

$$|P(z)| \leq M \exp\big(-nU(\mu, z) + nF\big), \qquad z \in \mathbf{C}. \tag{2.11}$$

In particular, (2.10) implies that

$$|w^n(z)P(z)| \leq M \qquad q.e. \text{ on } E. \tag{2.12}$$

(g) If

$$\mathcal{S}^* := \mathcal{S}^*(w, E) := \{z \in E : U(\mu, z) + Q(z) \leq F\}, \tag{2.13}$$

then $\mathcal{S}^* \supseteq \mathcal{S}$ is a compact set, Q is bounded on $\mathcal{S}^*$ and for any integer $n \geq 0$ and $P \in \Pi_n$,

$$||w^n P||_E = ||w^n P||_{\mathcal{S}^*}. \tag{2.14}$$

In fact, for any compact subset $K \subseteq E \backslash \mathcal{S}^*$, there are positive constants c_1, c_2 depending only on w, E, K such that for any integer $n \geq 1$ and $P \in \Pi_n$,

$$||w^n P||_K \leq c_1 \exp(-c_2 n)||w^n P||_{\mathcal{S}^*}. \tag{2.15}$$

The classical notions of capacity, transfinite diameter, Fekete and Chebyshev polynomials, Fekete points and Chebyshev constant can also be generalized. ([10]). For example, the *w-modified capacity* of a closed set E is defined by

$$cap(w,E) := \exp(-V(w,E)). \tag{2.16}$$

The *Fekete points* are the points $\{z^*_{kn} : \ k = 1,2,\cdots,n, \ \ n = 2,3,\cdots\}$ for which

$$\begin{aligned} \tau_n(w,E) &:= \max_{z_1,\cdots,z_n\in E} \prod_{1\le j<k\le n} \left\{|z_j - z_k| w(z_j) w(z_k)\right\}^{2/n(n-1)} \\ &= \prod_{1\le j<k\le n} \left\{|z^*_{jn} - z^*_{kn}| w(z^*_{jn}) w(z^*_{kn})\right\}^{2/n(n-1)} \end{aligned} \tag{2.17}$$

The *w-modified transfinite diameter* of the set E is defined by

$$\tau(w,E) := \inf_{n\ge 2} \tau_n(w,E). \tag{2.18}$$

An argument similar to that in ([2], P. 161) can be used (cf. [10]) to show not only that

$$cap(w,E) = \tau(w,E), \tag{2.19}$$

but that for any set of Fekete points $\{z^*_{kn} : \ k = 1,2,\cdots,n, \ \ n = 2,3,\cdots\}$ and any continuous, compactly supported function $g : E \to \mathbf{C}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} g(z^*_{kn}) = \int g\,d\mu, \tag{2.20}$$

where μ is the equilibrium measure defined in Theorem 2.1.

The notion of the Chebyshev constant can also be explored in this generality. We shall elaborate on this in the context of the L^p extremal polynomials. However, we note here the following simple consequence of (2.15) and (2.20).

PROPOSITION 2.2. ([10]) *If $K \subseteq E$ is compact, and for every sufficiently large integer n and polynomial $P \in \Pi_n$,*

$$||w^n P||_E = ||w^n P||_K, \tag{2.21}$$

then the support set S defined in Theorem 2.1 is a subset of K.

If $A \subseteq \mathbf{C}$ is a Borel set, σ is a positive Borel measure on A, $g : A \to \mathbf{C}$ is Borel measurable, then we define

$$||g||_{p,\sigma,A} := \begin{cases} \left(\int_A |g|^p d\sigma\right)^{1/p} & \text{if } 0 < p < \infty, \\ ||g||_A & \text{if } p = \infty. \end{cases} \tag{2.22}$$

The spaces $L^p(\sigma, A)$ are then defined as usual. We observe that our definition of $||g||_{\infty,\sigma,A}$

is not the usual one, but in the present context, the distinction would be inconsequential. Also, the fact that (2.22) does not define a norm if $0 < p < 1$ will be irrelevant for our purposes. Thus, we will continue to call the expressions defined in (2.22) 'norms' even in this case.

3. MAIN RESULTS.

Let $E \subseteq \mathbf{C}$ be closed, w be an admissible weight function on E, and σ be a fixed positive Borel measure defined on E. We shall assume that w is continuous on E and $[zw(z)]^n \in L^r(\sigma, E)$ for every r, $0 < r \leq \infty$. We also continue the notation described in Section 2.

In the sequel, we assume the notation and conditions of Theorem 2.1. The set E and the weight function w being fixed quantities, they will not be mentioned in the notations. We also adopt the following convention concerning constants. The symbols $c, c_1, c_2, \cdots$ will denote positive constants depending only on w, E and other explicitly prescribed parameters. Their value may not be the same in different occurrences of the same symbol, even within the same formula. Constants denoted by capital letters will retain their values after they are introduced.

DEFINITION 3.1. Let $K \subseteq E$ be compact, $0 < p \leq \infty$. We say that the $L^p(\sigma, E)$ norm of (w-) weighted polynomials *lives on* K if the following property holds. For every $\epsilon > 0$, there exists a bounded Borel set $\Delta \subseteq E$ with $\sigma(\Delta) < \epsilon$ and positive constants c_1, c_2 (depending only on w, E, p, ϵ) such that for every integer $n \geq 1$ and polynomial $P \in \Pi_n$,

$$||w^n P||_{p,\sigma,E} \leq \Big(1 + c_1 \exp(-c_2 n)\Big) ||w^n P||_{p,\sigma,K\cup\Delta} \tag{3.1}$$

Theorem 2.1 shows that the sup norm of weighted polynomials lives on $\mathcal{S}^*$. We shall show that for suitable measures σ, the $L^p(\sigma, E)$ norm also lives on $\mathcal{S}^*$. As mentioned in the introduction, a major issue here is the definition of a 'natural' measure in this case. We shall show that the following definition will work.

DEFINITION 3.2. The measure σ will be called *natural* if it satisfies each of the following conditions.

(M1) σ is a regular measure in the sense that for any Borel set $A \subseteq E$,

$$\begin{aligned} \sigma(A) &= \inf\{\sigma(O) : A \subseteq O, \quad O \text{ open } \} \\ &= \sup\{\sigma(K) : K \subseteq A, \quad K \text{ compact } \} \end{aligned} \tag{3.2}$$

(M2) For any compact subset K of E, $\sigma(K) < \infty$ and the restriction of σ to K has finite logarithmic energy.

(M3) There exists an integer $N \geq 0$ such that

$$\int_{|z|\geq 1} |z|^{-N} d\sigma(z) < \infty. \tag{3.3}$$

(M4) For $\delta > 0$ and $z \in \mathbf{C}$, let

$$\lambda_n(\sigma,\delta,z) := \min_{P\in\Pi_n} |p(z)|^{-2} \int_{E_\delta(z)} |P(t)|^2 d\sigma(t), \tag{3.4}$$

where

$$E_\delta(z) := \{t \in E : |t-z| \le \delta\}. \tag{3.5}$$

Then, for every $\delta > 0$ and every compact set $K \subseteq E$,

$$\limsup_{n\to\infty} ||\lambda_n^{-1}(\sigma,\delta,z)||_K^{1/n} \le 1. \tag{3.6}$$

We observe that none of the conditions in the above definition depends upon the weight function w. While the conditions (M1), (M2), (M3) above are fairly 'natural', perhaps the condition (M4) is not. The formula (3.4) defines a localized Christoffel function for σ. A good deal of research in the theory of orthogonal polynomials is devoted to obtaining the properties of σ from its Christoffel function ([7], [8]). In light of this research, the condition (M4) may be thought of as a density condition for σ on E. While it may not be very 'natural', it is satisfied in the important cases when E is a domain in C and σ is the two dimensional Lebesgue measure on E and when E is a sufficiently smooth curve and σ is its arc-length measure. In Theorem 3.5, we shall give some very general conditions, which are sometimes easier to verify, under which (M4) holds. Moreover, it will be apparent later (cf. Theorem 3.6) that (M4) is exactly the kind of condition which enables the $L^p(\sigma,E)$ norms to live on a fixed compact set.

We are finally in a position to formulate our main finite-infinite-range inequality.

THEOREM 3.3. *Let $E \subseteq$ C be closed, w be an admissible weight function on E and σ be a natural measure on E in the sense of Definition 3.2. Further suppose that w is continuous as a function on E and w^λ is also admissible on E for each $\lambda \in (0,1]$. Let $0 < p, r \le \infty$, and S^* be as defined in (2.13). Then, for any $\epsilon > 0$, there exists a bounded Borel set $\Delta := \Delta(p,r,\epsilon,w,E,\sigma) \subseteq E$ with $\sigma(\Delta) < \epsilon$ and positive constants c_1, c_2 depending only upon p,r,ϵ,w,E,σ, such that for any integer $n \ge 1$ and any polynomial $P \in \Pi_n$,*

$$||w^n P||_{r,\sigma,E\setminus(S^*\cup\Delta)} \le c_1 \exp(-c_2 n) ||w^n P||_{p,\sigma,S^*\cup\Delta}. \tag{3.7}$$

In particular, each of the $L^p(\sigma,E)$ norms of the w-weighted polynomials lives on S^*. Moreover, if K is any compact set, $0 < q \le \infty$ is a fixed number and the $L^q(\sigma,E)$ norm of the w-weighted polynomials lives on K, then $\sigma(S\setminus K) = 0$.

REMARK. In fact, it would be clear from our proof that the hypothesis in the last statement of Theorem 3.3 can be substituted by a slightly weaker hypothesis: For every $\epsilon > 0$, there exists a bounded Borel set Δ with $\sigma(\Delta) < \epsilon$ and

$$\limsup_{n\to\infty} \sup_{P\in\Pi_n} \left[\frac{||w^n P||_{q,\sigma,E\setminus(K\cup\Delta)}}{||w^n P||_{q,\sigma,K\cup\Delta}}\right]^{1/n} = 0. \tag{3.8}$$

A major step in the proof of Theorem 3.3 is to prove the following *Nikolskii-type inequalities* which are interesting in their own right.

THEOREM 3.4. *Under the hypothesis of Theorem 3.3, there exists a sequence of positive numbers* $\{N_n\}$ *such that*

$$\lim_{n\to\infty} N_n^{1/n} = 1 \tag{3.9}$$

and for $0 < p, r \le \infty$, *any integer* $n \ge 0$ *and* $P \in \Pi_n$,

$$\|w^n P\|_{r,\sigma,E} \le N_n^{|1/p-1/r|}\|w^n P\|_{p,\sigma,E} \tag{3.10}$$

Next, we discuss the condition (M4) in the Definition 3.2.

THEOREM 3.5. *Suppose that* $E \subseteq \mathbf{C}$ *be a closed regular set, i.e. for every* $R > 0$, *every point on the boundary of* $E \cap \{z : |z| \le R\}$ *is a regular point for the Dirichlet problem for each of the components of the complement of this set in the extended complex plane to which this point belongs. Let* σ *be a positive Borel measure on* E *which satisfies the conditions (M1), (M2), (M3) in Definition 3.2. Suppose that for every compact subset* $K \subseteq E$, *there exist positive constants* L, c *depending only on* E, K *and* σ *such that for every* $\delta > 0$,

$$a(\sigma, K, \delta) := \inf_{z\in K} \sigma\big(\{t \in E : |t - z| \le \delta\}\big) \ge c\delta^L. \tag{3.11}$$

Then σ *satisfies also the condition (M4) in Definition 3.2 and hence is a natural measure.*

We observe that the general Theorem 3.3 does not require any regularity conditions on E. The following theorem shows that the condition (M4) is actually necessary to achieve an inequality of the form (3.10).

THEOREM 3.6. *Suppose that* $E \subseteq \mathbf{C}$ *is a Borel set,* σ *is a positive Borel measure supported on* E *which satisfies conditions (M1), (M2), (M3) of Definition 3.2. If, for any compact set* $K \subseteq E$, *the inequality (3.10) holds with* $w \equiv 1$ *on* K, *then* σ *also satisfies the condition (M4) and hence, is a natural measure.*

Next, we give some applications of Theorem 3.3 to the theory of weighted extremal polynomials. Let $0 < p \le \infty$, E and w be fixed as before, and σ be a positive Borel measure on E. We define the extremal errors $\varepsilon_n(w, E, p, \sigma)$ and the extremal polynomials $T_n(w, E, p, \sigma; z) = z^n + \cdots \in \Pi_n$ by the formula

$$\varepsilon_n(w, E, p, \sigma) := \min_{P\in\Pi_n} \|w^n P\|_{\sigma,p,E} =: \|w^n T_n(w, E, p, \sigma)\|_{\sigma,p,E} \tag{3.12}$$

When $p = \infty$, then the extremal polynomials are the modified Chebyshev polynomials. For convenience, we omit the mention of p and σ from the notation for the extremal errors and polynomials in this case. Thus, for example, we write $\varepsilon_n(w, E) := \varepsilon_n(w, E, \infty, \sigma)$. The ideas in [5] and [11] can then be used to show that

$$\lim_{n\to\infty} \varepsilon_n^{1/n}(w, E) =: cheb(w, E) = \exp(-F), \tag{3.13}$$

where the constant $F := F(w, E)$ is definined in (2.6). When $p = 2$, then the extremal polynomial of degree n is precisely the orthogonal polynomial of degree n with respect to the weight function w^{2n} on E, normalized with leading coefficient 1. As a consequence of

Theorem 3.3 and the results in [6], we get the following

THEOREM 3.7. *Let E, w and σ be as in Theorem 3.3, $0 < p \le \infty$. Then*

$$\lim_{n\to\infty} \varepsilon_n^{1/n}(w, E, p, \sigma) = cheb(w, E), \tag{3.14}$$

where $cheb(w, E)$ is defined in (3.13). Moreover, if $\{\zeta_{1n}, \cdots, \zeta_{nn}\}$ are the zeros of $T_n(w, E, p, \sigma)$ and g is any compactly supported continous function on E, harmonic in the two dimensional interior of E, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} g(\zeta_{kn}) = \int g d\mu. \tag{3.15}$$

More general theorems similar to those in [6] with the sup norm replaced by the L^p norms can also be proved, but we omit these details in the interest of brevity. It is very easy to see that even in the simple case when $E = \{z : |z| \le 1\}$, $w \equiv 1$ on E and σ is the area measure, the hypothesis that g be harmonic in the interior of E cannot be dropped. The equation (3.13) is in fact, an equation for the balayages of the limiting measures for the zeros and the equilibrium measure μ. We do not wish to elaborate on this further here. The interested reader may refer to [6].

4. PROOFS.

The proof of Theorem 3.3 will be given in several stages. The first stage is to estimate the Christoffel functions associated with the weights $w^{2n}d\sigma$ (cf. (4.1) for a definition.) Next, we shall use these estimates to prove a Nikolskii-type inequality to compare different norms of a polynomial. This inequality will be used to prove that the norms actually live on a fixed compact set. This set will then be pruned to the desired set. We begin this program by recalling certain facts about the Christoffel functions.

If $A \subseteq \mathbf{C}$ is a Borel set, ν is a positive Borel measure on A having infinitely many points in its support and such that for each integer $m \ge 0$, $\int_A |z|^m d\nu < \infty$, then the Christoffel function is defined as follows. For any $z \in \mathbf{C}$ and integer $n \ge 1$,

$$\Lambda_n(\nu, A, z) := \inf_{P\in\Pi_n} |P(z)|^{-2} \int_A |P|^2 d\nu. \tag{4.1}$$

The following lemma summarizes some of the important properties of the Christoffel function which we shall need in the sequel. Let A and ν be as above, and $\{p_n(\nu, A, z) \in \Pi_n\}$ be the sequence of orthonormalized polynomials,

$$\int_A p_n(\nu, A, z)\overline{p_m(\nu, A, z)} d\nu(z) = \delta_{nm}, \qquad n, m = 0, 1, \cdots. \tag{4.2}$$

LEMMA 4.1. [12] *With the notation as in the previous paragraph,*

(a)

$$\Lambda_n^{-1}(\nu, A, z) = \sum_{k=0}^{n} |p_k(\nu, A, z)|^2 \ge 1. \tag{4.3}$$

(b) If $B \subseteq A$ then

$$\Lambda_n(\nu, B, z) \le \Lambda_n(\nu, A, z), \qquad z \in \mathbf{C}, \quad n = 1, 2, \cdots. \tag{4.4}$$

We now resume the notations and conventions adopted in Sections 2 and 3.

LEMMA 4.2. *Let σ be a natural measure on a closed set E, $K \subseteq E$ be compact. Then there exists a sequence of positive numbers $\{\delta_n\}$ such that $\delta_n \downarrow 0$ as $n \to \infty$ and*

$$\lim_{n\to\infty} ||\lambda_n^{-1}(\sigma, \delta_n, z)||_K^{1/n} = 1. \tag{4.5}$$

PROOF. In view of (4.3), it is enough to show that

$$\limsup_{n\to\infty} ||\lambda_n^{-1}(\sigma, \delta_n, z)||_K^{1/n} \le 1 \tag{4.6}$$

for a properly chosen sequence $\{\delta_n\}$. Using the condition (M4), for any integer $k \ge 1$, we may choose the smallest integer n_k such that

$$||\lambda_n^{-1}(\sigma, 1/k, z)||_K \le (1 + 1/k)^n, \qquad n \ge n_k. \tag{4.7}$$

With the sequence $\{\delta_n\}$ defined by

$$\delta_n := 1/k, \qquad n_k \le n < n_{k+1}, \quad k = 1, 2, \cdots,$$

and $\delta_n := 1$ for $1 \le n \le n_k$, it is readily seen from (4.7) that (4.6) is satisfied. ■

LEMMA 4.3. *There exists a sequence $\{M_n := M_n(w, E, \sigma)\}$ of positive numbers such that*

$$||w^{2n}(z)\Lambda_n^{-1}(w^{2n}d\sigma, E, z)||_E \le M_n \tag{4.8}$$

and

$$\lim_{n\to\infty} M_n^{1/n} = 1. \tag{4.9}$$

We note that in (4.8), the integer n appears several times. This notation will be consistently used. Thus, for instance, if $k \ge 1$ is another integer, then

$$||w^{2nk}(z)\Lambda_{nk}^{-1}(w^{2nk}d\sigma, E, z)||_E \le M_{nk}.$$

PROOF OF LEMMA 4.3. Without loss of generality, we may assume that $E \subseteq$ *supp*(w). Also, it is obvious that we need to define M_n only for large value of n. For $\delta > 0$, set

$$\omega(\delta) := \sup\{|Q(z) - Q(t)| : z \in E, t \in \mathcal{S}^*, |z - t| \le \delta\}. \tag{4.10}$$

Since w is continuous on E and bounded from below by a positive constant on the compact set $\mathcal{S}^*$, it is easy to see that $\omega(\delta) \downarrow 0$ as $\delta \to 0+$. Let δ_n be the sequence defined by Lemma 4.2 for the compact set $\mathcal{S}^*$ and n be large enough so that $\omega(\delta_n) < \infty$. Let $z \in \mathcal{S}^*$ and $P \in \Pi_n$. Then

$$\int |P(t)|^2 w^{2n}(t)d\sigma(t) \geq w^{2n}(z)\exp(-2n\omega(\delta_n))\int_{\{t\in E:|t-z|\leq\delta_n\}} |P(t)|^2 d\sigma(t) \geq w^{2n}(z)|P(z)|^2\exp(-2n\omega(\delta_n))\lambda_n(\sigma,\delta_n,z) \tag{4.11}$$

where $\lambda_n(\sigma,\delta_n,z)$ is defined in (3.4). Let

$$M_n := (n+1)\exp(2n\omega(\delta_n))||\lambda_n^{-1}(\sigma,\delta_n,z)||_{S^*}. \tag{4.12}$$

Then (4.11) and (4.3) yield that for $z \in S^*$,

$$|w^{2n}(z)p_k^2(w^{2n}d\sigma, E, z)| \leq M_n/(n+1), \quad k = 0,\cdots,n. \tag{4.13}$$

In view of Theorem 2.1(g), the inequality (4.13) persists for $z \in E$. Using (4.3) again, we get (4.8). The estimates (4.8) and (4.3) imply that

$$\liminf_{n\to\infty} M_n^{1/n} \geq 1.$$

Since $\omega(\delta_n) \to 0$ as $n \to \infty$, (4.5) and (4.12) imply (4.9). ■

Lemma 4.3 completes the first step in our proof of Theorem 3.3. The following lemma gives the Nikolskii-type inequalities 'in one direction'. This will be enough for our purposes.

LEMMA 4.4. *Let $n \geq 1$ be an integer, $P \in \Pi_n$, $0 < p < r \leq \infty$. Let k be the least positive integer with $2k \geq p$, and $\{M_n\}$ be the sequence defined in Lemma 4.3. Then,*

$$||w^n P||_{r,\sigma,E} \leq M_{kn}^{(1/p-1/r)}||w^n P||_{p,\sigma,E}. \tag{4.14}$$

PROOF. We may write (cf. [12])

$$P^k(z) = \int P^k(t)K_{kn}(z,t)w^{2nk}(t)d\sigma(t), \tag{4.15}$$

where, in this proof only,

$$K_m(z,t) := \sum_{l=0}^{m} p_l(w^{2nk}d\sigma, E, z)\overline{p_l(w^{2nk}d\sigma, E, t)}. \tag{4.16}$$

Using Cauchy-Schwartz inequality in (4.15), and taking into account (4.3), (4.8) we get

$$||w^n P||_E \leq M_{kn}^{1/2k}||w^n P||_{2k,\sigma,E}. \tag{4.17}$$

Using a convexity argument repeatedly, we get first

$$||w^n P||_E \leq M_{kn}^{1/p}||w^n P||_{p,\sigma,E} \tag{4.18}$$

and then (4.14). ■

The next step in the proof of Theorem 3.3 is to show that the L^p-norm lives on some compact set.

LEMMA 4.5. *Let $0 < p < \infty$. Then there exists $R > 0$ such that, with $K := \{z \in E : |z| \le R\}$, we have, for every integer $n \ge 1$ and $P \in \Pi_n$,*

$$||w^n P||_{p,\sigma,E\setminus K} \le c_1 \exp(-c_2 n)||w^n P||_{p,\sigma,K} \tag{4.19}$$

where c_1, c_2 are positive constants depending only on p, σ, w and E.

PROOF. Let k be the least positive integer for which $2k \ge p$ and $n \ge N+1$, where N is defined in condition (M3) in Definition 3.2. Since $w^{n/(n+1+\lfloor n/p \rfloor)}$ is admissible, there exists a positive number A such that, for $z \in E$,

$$\begin{aligned} |z^{\lfloor n/p\rfloor+1}P(z)w^n(z)| &\le ||\zeta^{\lfloor n/p\rfloor+1}P(\zeta)w^n(\zeta)||_{E\cap\{|\zeta|\le A\}} \\ &\le A^2 A^{n/p}||w^n P||_E \\ &\le A^2 A^{n/p} M_{kn}^{1/p}||w^n P||_{p,\sigma,E}. \end{aligned} \tag{4.20}$$

(Here, we used Lemma 4.4 in the last inequality.) Hence, if $B > A$,

$$\int_{|z|\ge B, z\in E} |P(z)|^p w^{np}(z) d\sigma(z) \le A^{2p} B^N A^n M_{kn} B^{-n} \int_{|z|\ge B, z\in E} |z|^{-N} d\sigma(z). \tag{4.21}$$

In view of (4.9), it is now easy to deduce (4.19). ■

PROOF OF THEOREM 3.3. Let $\epsilon > 0$ and K be the compact set introduced in Lemma 4.5. Without loss of generality, we may assume that $\mathcal{S}^* \subseteq K$. Since σ is regular, there exists a bounded open set U such that $\mathcal{S}^* \subseteq U$ and $\sigma(cl(U)\setminus\mathcal{S}^*) < \epsilon$, where $cl(U)$ denotes the closure of U. In view of Theorem 2.1(g), there exists a positive constant c, independent of n and P, such that

$$|w^n(z)P(z)| \le \exp(-cn)||w^n P||_E, \qquad z \in K\setminus U. \tag{4.22}$$

Using Lemma 4.4 and (4.9), this leads to

$$||w^n P||_{p,\sigma,K\setminus U} \le \exp(-cn)||w^n P||_{p,\sigma,E}, \tag{4.23}$$

where c is another positive constant. It follows from (4.19) and (4.23) that

$$||w^n P||_{p,\sigma,E\setminus U} \le c_1 \exp(-c_2 n)||w^n P||_{p,\sigma,E} \tag{4.24}$$

for some positive constants c_1, c_2. Since $\sigma(cl(U)\setminus\mathcal{S}^*) < \epsilon$, we have proved that the $L^p(\sigma, E)$ norm of the weighted polynomials lives on $\mathcal{S}^*$.

It is now simple to see that the estimate (4.14) can be extendend to

$$||w^n P||_{r,\sigma,E} \le N_n^{|1/p-1/r|}||w^n P||_{p,\sigma,E}, \qquad 0 < p, r \le \infty, \tag{4.25}$$

where N_n is a sequence of positive numbers with $\lim_{n\to\infty} N_n^{1/n} = 1$. If we now use (4.24) with r in place of p, and take (4.25) into account, then it is easy to prove (3.7). To prove the second part of Theorem 3.3, let $\epsilon > 0$ be arbitrary. We find a bounded Borel set Δ

with $\sigma(\Delta) < \epsilon$ such that (3.8) holds. In view of (4.25), applied once to $E\backslash(K \cup \Delta)$ and once to E, we see that for sufficiently large integer n and $P \in \Pi_n$,

$$||w^n P||_E = ||w^n P||_{K\cup\Delta}.$$

Proposition 2.2 then yields that $\mathcal{S} \subseteq K \cup \Delta$. Since $\sigma(\Delta) < \epsilon$ and ϵ was arbitrary, this proves that $\sigma(\mathcal{S}\backslash K) = 0$. ∎

We observe that the proof of Theorem 3.3 includes a proof of Theorem 3.4. The proofs of Theorems 3.5 and 3.6 use the following lemma.

LEMMA 4.6. *Let $E \subseteq \mathbf{C}$ be a Borel set, ν be a measure supported on E, for every compact set $K \subseteq E$ and $\eta > 0$, there exist a seqence of positive numbers $L_n := L_n(\nu, K, E, \eta)$ with*

$$\limsup_{n\to\infty} L_n^{1/n} \le 1, \tag{4.26}$$

and the Christoffel functions defined in (4.1) satisfy

$$\Lambda_n^{-1}(\nu, K_\eta, z) \le L_n, \qquad z \in K, \tag{4.27}$$

where

$$K_\eta := \{z : |z - t| \le \eta \text{ for some } t \in K\}. \tag{4.28}$$

Then ν satisfies the condition (M4) in Definition 3.2.

REMARK. In [9], a measure σ with compact support A is said to be *completely regular* if

$$\lim_{n\to\infty} ||p_n(\sigma, A, z)||_A^{1/n} = 1.$$

When σ is not compactly supported, but the restriction of σ to every compact set is completely regular in this sense, then the Lemma shows that σ satisfies condition (M4) of Definition 3.2.

PROOF. Let $\delta > 0$, and $K \subseteq E$ be a fixed compact set. With the notation as in (3.5), we find points $z_1, \cdots z_m$ in K such that $K \subseteq \bigcup_{k=1}^{m} E_{\delta/4}(z_k)$. The hypotheses of Lemma 4.6 applied to each $E_{\delta/4}(z_k)$ shows that for any $\epsilon > 0$, there exists an integer N such that

$$\Lambda_n^{-1}(\nu, E_{\delta/2}(z_k), z) \le (1+\epsilon)^n, \qquad z \in E_{\delta/4}(z_k), \quad k = 1, \cdots, m, \quad n \ge N. \tag{4.29}$$

We observe that N depends only on K, δ and ν. Now, if $z \in K$ is arbitrary, then $E_\delta(z) \supseteq E_{\delta/2}(z_k)$ for some k, $1 \le k \le m$. Lemma 4.1(b) then implies that

$$\Lambda_n^{-1}(\nu, E_\delta(z), z) \le (1+\epsilon)^n, \qquad z \in K, \quad n \ge N. \tag{4.30}$$

This gives (3.6) with ν in place of σ as required. ∎

It is convenient to prove Theorem 3.6 first.

PROOF OF THEOREM 3.6. The hypothesis of Theorem 3.6 imply, with $r = \infty$ and $p = 2$ in (3.10), that

$$\Lambda_n^{-1}(\sigma, K, z) \le cN_n^{1/2} \qquad z \in K, \tag{4.30}$$

In view of Lemma 4.1 (b), this implies (4.27) with σ in place of ν. Lemma 4.6 then completes the proof. ■

In the proof of Theorem 3.5, we need another lemma to estimate the derivative of a polynomial in terms of its maximum modulus on K. Lemma 4.7 is most probably not new, but we include a proof because it is very simple.

LEMMA 4.7. *Let K be a regular (in the sense described in the statement of Theorem 3.5), compact set, and let ν be its classical equilibrium measure (cf. [13], P. 55). We set*

$$u(z) := u(K, z) := \int \log(1/|z-t|)d\nu(t), \qquad z \in \mathbf{C}. \tag{4.31}$$

Then u is continuous on $\mathbf{C}$. Let V be the convex hull of the set $\{z : |z-t| \le 1$ for some $t \in K\}$, and

$$\Omega(\delta) := \max\{|u(z) - u(t)| : z, t \in V, |z-t| \le \delta\}. \tag{4.32}$$

If n is any positive integer, $P \in \Pi_n$, and $0 < \delta < 1$ is arbitrary, then

$$|P'(z)| \le \delta^{-1} \exp\Big(n(\Omega(\delta) - u(z) - \log cap(K))\Big)||P||_K. \tag{4.33}$$

PROOF. The fact that u is continuous is well known. (cf. [13]) The Bernstein-Walsh inequality ([14], P.77) yields

$$|P(\zeta)| \le \exp(-nu(\zeta) - \log cap(K))||P||_K, \qquad \zeta \in \mathbf{C}. \tag{4.34}$$

If $z \in \mathbf{C}$ and $\Gamma := \{\zeta : |\zeta - z| = \delta\}$, then

$$P'(z) = \frac{1}{2\pi i}\int_\Gamma \frac{P(\zeta)}{(\zeta - z)^2}d\zeta. \tag{4.35}$$

The estimate (4.33) follows easily from (4.35), (4.34) and (4.32). ■

We are now in a position to prove Theorem 3.5.

PROOF OF THEOREM 3.5. Since E is regular, every compact subset of E is contained in a regular compact subset of E. In view of Lemma 4.1 (b), it is therefore enough to verify (3.6) only in the case when K is regular. Let K be a fixed, regular compact subset of E, and we adopt the notation in Lemma 4.7. Let $n \ge 1$ be any integer, $P \in \Pi_n$ and $||P||_K = 1$. Let $z_0 \in K$ be found so that

$$|P(z_0)| = ||P||_K = 1. \tag{4.36}$$

Let

$$\epsilon_n := (2n)^{-1}\exp(-2n\Omega(1/n)). \tag{4.37}$$

Then, for $z \in V$, $|z - z_0| \le \epsilon_n$ $(\le 1/n)$, Lemma 4.7 yields with $1/n$ in place of δ,

$$|P(z) - P(z_0)| \le \epsilon_n \max\{|P'(\zeta)| : |\zeta - z_0| \le \epsilon_n\} \le 1/2\exp(-nu(z_0) - \log cap(K)). \tag{4.37}$$

Since z_0 is a regular point of K, this and (4.36) imply that

$$|P(z)| \geq 1/2, \qquad z \in V, \quad |z - z_0| \leq \epsilon_n. \tag{4.38}$$

Since $\epsilon_n \to 0$ as $n \to \infty$, we may choose n so large that $\epsilon_n < \eta$. We integrate both sides of (4.38) with respect to σ and use (3.11) to get

$$\int_{K_\eta} |P(z)|^2 d\sigma(z) \geq a(\sigma, K, \epsilon_n)||P||_K \geq c\epsilon_n^L||P||_K. \tag{4.39}$$

Since $\epsilon_n^{-L/n} \to 1$ as $n \to \infty$, we have shown that σ satisfies the hypotheses of Lemma 4.6. In view of that Lemma, the proof is now complete. ∎

Theorem 3.7 follows from Theorem 3.4 and the known results about the asymptotically extremal polynomials in the sup norm sense. (cf. (3.13) and [11]).

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