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RESEARCH NOTES

ON MULTIPARAMETER SPECTRAL THEORY

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ABSTRACT. A connection between the numerical range of the multiparameter linear system $P(\lambda)$ and the joint numerical range of the (commuting) separating operator system is given.

KEY WORDS AND PHRASES. Multiparameter operator system, uniform reasonable cross-norm, numerical range, and spectral problems.

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1. INTRODUCTION. We consider the following multiparameter linear operator system:

$$P(\lambda) = (P_1(\lambda), ..., P_n(\lambda)),$$

where

$$P_1(\lambda) = B_1 - \lambda_1 A_{11} - \cdots - \lambda_n A_{1n},$$
$$P_2(\lambda) = B_2 - \lambda_1 A_{21} - \cdots - \lambda_n A_{2n},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$P_n(\lambda) = B_n - \lambda_1 A_{n1} - \cdots - \lambda_n A_{nn},$$

and the operators B_j and A_{jk} acting on different (smooth) complex Banach spaces X_j are assumed to be bounded and linear $(j, k = 1, ..., n)$.

DEFINITION 1.1. Let X_1 be a (smooth) complex Banach space. Then in this smooth space there is a unique map $\phi: X_1 \to X_1^*$ such that (see [1]).

$$\| \phi(x_1) \| = \| x_1 \|, \quad < x_1, \phi(x_1) > = \| x_1 \|^2, \quad \text{for } x_1 \in X_1.$$

Now using the function ϕ, we can define a semi-inner product on X_1 by

$$[x_1, y_1]_1 = < x_1, \phi(y_1) >_1, \quad x_1, y_1 \in X_1.$$

DEFINITION 1.2. Let $\hat{X} = X_1 \hat{\otimes}_\alpha ... \hat{\otimes}_\alpha X_n$ denote the completion of the tensor product $X_1 \otimes ... \otimes X_n$ with respect to a uniform reasonable cross-norm α. For details, see [2]. Let $[.,.]_j$ be a semi-inner product on a Banach space X_j with respect to a norm α_j. Then there is a semi-inner product $[.,.]$ on $\hat{X}$, defined by

$$[x, y] = [x_1, y_1]_1 \cdot \cdot [x_n, y_n]_n,$$

where $x = x_1 \otimes ... \otimes x_n \in \hat{X}$, and $y = y_1 \otimes ... \otimes y_n \in \hat{X}$.

Let $P_j(\lambda_1, ..., \lambda_n)$ be bounded linear operators on $X_j, 1 \leq j \leq n$.

DEFINITION 1.3. We define the (spatial) numerical range of the system $P(\lambda)$ as the set

$$V[P_1(\lambda),\dots,P_n(\lambda)] = \bigcap_{j=1}^{n} \bigcup_{0 \neq x_j \in X_j} \{(\lambda_1,\dots,\lambda_n) \in C^n : [P_j(\lambda)x_j, x_j]_j = 0\},$$

where $[x_j, x_j] = 1$.

We next introduce the operators which arise in the separation of the spectral parameters.

To each operator $T_j: X_j \to X_j$, we associate an operator acting on $\widehat{X}$ given by

$$T_j^{\otimes} = I_1 \otimes \dots \otimes I_{j-1} \otimes T_j \otimes I_{j+1} \otimes \dots \otimes I_n.$$

In order to study the linear operator system $P(\lambda)$, we construct the operators $\Delta_0, \Delta_1, \dots, \Delta_n$, which are well-defined determinants of the operator matrix $(A_{ij}^{\otimes})_{i,j=1}^{n}$ and the matrices obtained from this matrix by replacing the $j-th$ column by the column of operators $B_1^{\otimes},\dots,B_n^{\otimes}$, $j = 1,\dots,n$. For details, see [3,4].

As in the multiparameter spectral theory, the separation of the spectrum is understood to mean the reduction of the spectral problems for the system $P(\lambda)$ to the spectral problems for the commuting operator system $(\Delta_0^{-1}\Delta_1 - \lambda_1 I, \dots, \Delta_0^{-1}\Delta_n - \lambda_n I)$, the system $(\Delta_0^{-1}\Delta_j - \lambda_j I)$, $1 \leq j \leq n$, is called a separating system.

In the present problem, we consider the case when the operator Δ_0 is invertible, and the separating system $(\Delta_0^{-1}\Delta_j - \lambda_j I)$ is commutative $(1 \leq j \leq n)$. The aim of this note is to announce the following result on the multiparameter spectral theory.

2. MAIN RESULTS.

Let $(\Delta_0^{-1}\Delta_1 - \lambda I, \dots, \Delta_0^{-1}\Delta_n - \lambda_n I)$ be a commutative family of operators acting on $\widehat{X}$.

DEFINITION 2.1. We introduce the concept of the joint (spatial) numerical range of the operator system $(\Delta_0^{-1}\Delta_j - \lambda_j I)$, $1 \leq j \leq n$, as the set

$$V_{\otimes}[\Delta_0^{-1}\Delta_1 - \lambda I, \dots, \Delta_0^{-1}\Delta_n - \lambda_n I] = \{\lambda = (\lambda_1,\dots,\lambda_n) \in C^n : [\Delta_0^{-1}\Delta_j x, x] = \lambda_j [x,x]\},$$

for $x = x_1 \otimes \dots \otimes x_n \in \widehat{X}$, and $[x,x] = 1$.

We now write the result connecting the numerical range of the system $P(\lambda)$ and the joint numerical range of the (commuting) separating operator system $(\Delta_0^{-1}\Delta_j - \lambda_j I)$, $1 \leq j \leq n$.

THEOREM 2.1. $V[P_1(\lambda),\dots,P_n(\lambda)] = V_{\otimes}[\Delta_0^{-1}\Delta_1 - \lambda_1 I, \dots, \Delta_0^{-1}\Delta_n - \lambda_n I]$.

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