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SEMI-TOPOLOGICAL PROPERTIES

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ABSTRACT. A property preserved under a semi-homeomorphism is said to be a semi-topological property. In the present paper we prove the following results: (1) A topological property P is semi-topological if and only if the statement '$(X,\mathcal{T})$ has P if and only if $(X,F(\mathcal{T}))$ has P' is true where $F(\mathcal{T})$ is the finest topology on X having the same family of semi-open sets as $(X,\mathcal{T})$. (2) If P is a topological property being minimal P is semi-topological if and only if for each minimal P space $(X,\mathcal{T})$, $\mathcal{T} = F(\mathcal{T})$.

KEY WORDS. Semi-open sets, Semi-continuous, semi-homeomorphism, semi-topological, minimal P-spaces.
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1. INTRODUCTION. In 1963, Levine [1] introduced the concept of a semi-open set. A set $A \subseteq X$ is said to be semi-open if there exists an open set $U \subseteq X$ such that $U \subseteq A \subseteq cl\, U$ where cl U denotes the closure of U. In 1972, Crossley and Hildebrand [2] talked about the concept of a semi-homeomorphism. A property preserved under a semi- homeomorphism is said to be a semi-topological property. Every semi- topological property is a topological property. Crossley and Hildebrand [2] proved that some topological properties are in fact semi-topological. In the present paper we propose to develop a technique which enables one to establish whether a topological property is semi-topological or not. We also develop a technique to identify those topological properties P for which being minimal P is not semi-topological. These techniques are used to show that the property of being an s-Urysohn space [3] is semi-topological. Also we show that the properties of being a completely s-regular space [4], a semi-T_D-space [5], a T_D-space [6], a semi-normal space [7] and a completely semi-normal space [8] are not semi-topological. Furthermore, for P = Hausdorff, sequential or completely regular we prove that being minimal P is not semi-topological.

2. THE MAIN RESULT.

Crossley and Hildebrand [2] showed that two different topologies on a set X can have the same family of semi-open sets. However, they proved that for any space $(X,\mathcal{T})$, there exists the finest topology $F(\mathcal{T})$ for X having the same family of semi-open sets as $(X,\mathcal{T})$. Crossley [9] proved that this topology is characterized as $F(\mathcal{T}) = \{U - N: U \in \mathcal{T}$ and N is a nowhere dense subset of $(X,\mathcal{T})\}$. In fact the topology $F(\mathcal{T})$ is the same as the topology $\mathcal{T}^{\alpha}$ of Njåstad [10]. The topology $\mathcal{T}^{\alpha}$ consists of all α-sets of a space $(X,\mathcal{T})$ where a set A is said to be an α-set if $A \subseteq \text{int}(\text{cl}(\text{int } A))$ [10].

Before we state our main result we shall give some definitions.

DEFINITION 2.1. A function $f: X \to Y$ is said to be <u>semi- continuous</u> [1] (respectively, <u>irresolute</u> [2]) if the inverse image of every open (respectively, semi-open) set is semi-open. $f: X \to Y$ is said to be <u>pre-semi-open</u> [2] if the image of every semi-open set is semi-open. A one-one, onto, irresolute and pre-semi-open function is said to be a <u>semi-homeomorphism</u> [2].

THEOREM 2.2. A topological property P is a semi-topological property if and only if the statement 'A space $(X,\mathcal{T})$ has P if and only if $(X,F(\mathcal{T}))$ has P' is true.

PROOF. Suppose P is a semi-topological property. The identity function $f: (X,\mathcal{T}) \to (X,F(\mathcal{T}))$ is a semi-homeomorphism. Hence if $(X,\mathcal{T})$ has P, $(X,F(\mathcal{T}))$ has P. On the other hand, if $(X,F(\mathcal{T}))$ has P, then the identity function $g: (X,F(\mathcal{T})) \to (X,\mathcal{T})$ is a semi-homeomorphism and so (X,T) will have P.

Conversely, let the statement '$(X,\mathcal{T})$ has P if and only if $(X,F(\mathcal{T}))$ has P' be true. Let $f: (X,\mathcal{T}) \to (Y,\mathcal{U})$ be a semi-homeomorphism and let $(X,\mathcal{T})$ have P. Then $(X,F(\mathcal{T})$ has P, in view of the hypothesis. In view of Theorem 2.6 in [2], $(Y,F(\mathcal{U}))$ has the property P and so $(Y,\mathcal{U})$ has P.

3. APPLICATIONS.

DEFINITION 3.1 [3]. A space X is said to be <u>s-Urysohn</u> if for any two distinct points x and y there exist semi-open sets U and V such that $x \in U$, $y \in V$ and $\text{cl}U \cap \text{cl}V = \emptyset$.

THEOREM 3.2. A space $(X,\mathcal{T})$ is s-Urysohn if and only if $(X,F(\mathcal{T}))$ is s-Urysohn.

PROOF. Easy proof of the 'only if' part is omitted. To prove the 'if' part, let $(X,F(\mathcal{T}))$ be s-Urysohn and let x and y be any two distinct points of X. Then there exist semi-open sets U and V such that $x \in U$, $y \in V$ and $\text{cl}_{F(\mathcal{T})}(U) \cap \text{cl}_{F(\mathcal{T})}(V) = \emptyset$. Suppose there exists a point $p \in \text{cl}_{\mathcal{T}}(U) \cap \text{cl}_{\mathcal{T}}(V)$. In that case we shall prove that $p \in \text{cl}_{F(\mathcal{T})}(U) \cap \text{cl}_{F(\mathcal{T})}(V)$. Let W be an $F(\mathcal{T})$-open set containing p. Hence there exists a nowhere dense subset M of $(X,\mathcal{T})$ such that $W \cup M \in T$ and $W \cap M = \emptyset$. Now, $U \cap W = (U \cap (W \cup M)) - M$. Also $U \cap (W \cup M)$, being the intersection of a $\mathcal{T}$-open set and a semi-open set, is semi-open. Also $\text{cl}_{\mathcal{T}}(X - M) = X$. Hence $s\text{-cl}(X - M) = X$ [11]. Therefore, $(U \cap (W \cup M)) \cap (X - M) \neq \emptyset$. In other words, $U \cap W \neq \emptyset$ which implies that $p \in \text{cl}_{F(\mathcal{T})}(U)$. Similarly we can prove that $p \in \text{cl}_{F(\mathcal{T})}(V)$, which is a contradiction. Therefore, $\text{cl}_{\mathcal{T}}(U) \cap \text{cl}_{\mathcal{T}}(V) = \emptyset$. Hence $(X,\mathcal{T})$ is s-Urysohn.

DEFINITION 3.3 [4]. A space $(X,\mathcal{T})$ is said to be <u>completely s-regular</u> if for any closed set F and a point $x \notin F$, there exists a real valued semi-continuous function $f: X \to [0,1]$ such that $f(x) = 0$ and $f(F) = 1$.

THEOREM 3.4. Complete s-regularity is not semi-topological.

PROOF. Let βN be the Stone-Cech compactification of the discrete set of positive integers N. If $m \in \beta N - N$, then $\beta N - (N \cup \{m\})$ and $\{m\}$ are disjoint semi-closed subsets of βN. Let P and Q be semi-open subsets of βN such that $\beta N - (N \cup \{m\}) \subseteq P$ and $m \in Q$, and let V,W be open subsets such that $V \subseteq P \subseteq cl\ V$, $W \subseteq Q \subseteq cl\ W$. N is a dense subset of the extremally disconnected space βN: hence cl W is open and uncountable. Thus $cl\ W \cap cl\ V \neq \phi$ and $\phi \neq W \cap V \subseteq P \cap Q$. Hence there is no semi-continuous $g: \beta N \to [0,1]$ such that $g(\beta N - (N \cup \{m\})) = \{1\}$ and $g(m) = 0$. Let $\mathcal{T}$ denote the topology for the Stone-Cech compactification. Then $F(\mathcal{T}) = \{V - M: V \in \mathcal{T}$ and M nowhere dense in $\beta N\}$. $(\beta N, F(\mathcal{T}))$ has the same collection of semi-open subsets as $(\beta N, \mathcal{T})$. $\beta N - (N \cup \{m\})$ is nowhere dense in βN and hence $F(\mathcal{T})$ - closed. Therefore $(\beta N, F(\mathcal{T}))$ is not completely s-regular and hence in view of the main result complete s-regularity is not semi-topological.

The proof of the following theorem is easy and hence omitted.

THEOREM 3.5. A space $(X,\mathcal{T})$ is semi-T_0 [12] (respectively, semi-T_1 [12], semi-T_2 [12], semi-US [13], semi-Urysohn [3], s-normal [14], completely s-normal [8], hyper-connected [15], S-closed [16], strongly S-closed [17], a Baire space) if and only if $(X,F(\mathcal{T}))$ is semi-T_0 (respectively, semi-T_1, semi-T_2, semi-US, semi-Urysohn, s-normal, completely s-normal, hyper-connected, S-closed, strongly S-closed, a Baire space).

As an application of our main result, we can now state the following, in view of Theorems 3.2 and 3.5.

THEOREM 3.6. The following properties are semi-topological (1) semi-T_0 (2) semi-T_1 (3) semi-T_2 (4) semi-US (5) semi-Urysohn (6) s-Urysohn (7) s-normal (8) hyperconnected (10) S-closed (11) strongly S-closed (11) Baire Space.

REMARK 3.7. Some of the properties mentioned in Theorem 3.6 are in fact preserved under functions weaker than a semi-homeomorphism [3,14,17,18,19,20,21]. It may be noted that a similar method is used to prove that the properties of being T_2, being strongly Hausdorff and of being Urysohn are semi-topological [2,22].

Crossley and Hildebrand [2] proved that the properties T_0, T_1, T_3, T_4, T_5, regularity, normality, complete normality, paracompactness, Lindelöfness and metrizability are not semi- topological. Theorem 3.4 proves that property of being completely s-regular is not semi-topological. We shall now show that the properties of being semi-T_D, T_D, semi-normal and completely semi-normal, which are defined below, are not semi-topological.

DEFINITION 3.8. A space X is said to be <u>semi - T_D</u> [5] (respectively a <u>T_D - space</u> [6]) if for every $x \in X$, the derived set $d\{x\}$ of $\{x\}$ is semi-closed (respectively, closed). A space X is said to be <u>semi-normal</u> respectively <u>completely semi-normal</u> [8] if for any two disjoint closed sets A and B (respectively, for any two subsets A and B of X such that $clA \cap B = \emptyset = A \cap clB$) there exist disjoint semi-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

It may be noted that semi-normal spaces have been studied by Maheshwary and

Prasad [7] under the name s-normal.

EXAMPLE 3.9. Let $X = [-1,1] - \{0\}$ and let $\mathcal{T}$ be generated by the following collection of basic open sets: $\{X, \emptyset, \{-1\}, \{1\}, \{1,-1\}, [-1,0[,]0,1]\}$ where $]0,1]$ denotes the interval open at 0 and closed at 1. Then $(X,\mathcal{T})$ is not semi - T_D, but $(X,F(\mathcal{T}))$ is semi - T_D. For, let $k \in X$. Suppose that $-1 < k < 0$. Then for $-1 < a < k$, $[-1,a]$ is an $F(\mathcal{T})$-open set containing a but not k. Again for $k < a < 0$, $[-1,k[\cup]k,a]$ is an $F(\mathcal{T})$-open set containing a but not k. Also for $0 < a < k < 1$, $[a,k[\cup]k,1]$ is an $F(\mathcal{T})$-open set containing a but not k. Similarly, if $k < a < 1$, then $[a,1]$ is an $F(\mathcal{T})$-open set containing a. Thus for $k \in]-1,0[\cup]0,1[$, $d\{k\} = \emptyset$ and hence semi-closed. Now suppose that $k = -1$. Then $cl_{F(\mathcal{T})}(\{k\}) = [-1,0[$. Hence $d\{k\} =]-1,0[= X - \{]0,1] \cup \{-1\}\}$. Similarly, we can prove that $d\{1\}$ is also semi-closed. Thus $(X,F(\mathcal{T}))$ is semi - T_D. But, since $(X,\mathcal{T})$ is not a semi - T_D - space, it follows that the property of being a semi- T_D - space is not semi-topological.

REMARK 3.10. It may be noted that $(X,F(\mathcal{T}))$ of the space considered in Example 3.9 is in fact a T_D - space. Hence, it shows that the property being a T_D - space is not semi-topological. However, we shall give another example to show that the concept of a T_D - space is not semi-topological.

EXAMPLE 3.11. Let $X = [0,1]$ and $\mathcal{T} = \{\emptyset, X\} \cup \{[0, \frac{1}{2^n}]$, $n = 1,2,3,\ldots\ldots\ldots\}$. It can be verified that $(X,\mathcal{T})$ is not a T_D - space. We shall prove that $(X,F(\mathcal{T}))$ is a T_D - space. Let $b \in X$ such that $\frac{1}{2^{n+1}} < b < \frac{1}{2^n}$ for some n. For each $a \in X$ such that $\frac{1}{2^{n+1}} < a < b$, $[0, \frac{1}{2^{n+1}}] \cup \{a\}$ is an $F(\mathcal{T})$-open set containing a but not b. If $b < a < \frac{1}{2^n}$, then also, $[0, \frac{1}{2^{n+1}}] \cup \{a\}$ is an $F(\mathcal{T})$-open set containing a having empty intersection with $\{b\}$. Similarly we can find an $F(\mathcal{T})$-open set U containing a but not b for each $a \in X$ where $a \neq b$. Therefore $d\{b\} = \emptyset$. Hence the property of being a T_D - space is not semi-topological.

REMARK 3.12. The above example also shows that the property of being a T_o - space is not semi-topological since $(X,\mathcal{T})$ is not T_o.

EXAMPLE 3.13. Let $X = \{a,b,c\}$ and $\mathcal{T} = \{\emptyset, X, \{a\}, \{a,c\}\}$. Then $F(\mathcal{T}) = \{\emptyset, X, \{a\}, \{a,b\}, \{a,c\}\}$. $(X,\mathcal{T})$ is semi-normal but $(X,F(\mathcal{T}))$ is not semi-normal.

REMARK 3.14. The above example also shows that the property of being a completely semi-normal space is not semi-topological.

It is well known that for a topological property P, if $(X,\mathcal{T})$ is minimal P and $f: (X,\mathcal{T}) \to (Y,\mathcal{U})$ is a homeomorphism, then $(Y,\mathcal{U})$ is minimal P. However as we shall soon see being minimal P need not be a semi-topological property. The following result provides a technique to identify those topological properties P for which being minimal P is not semi-topological.

THEOREM 3.15. If P is a topological property being minimal P is semi-topological if and only if for each minimal P space $(X,\mathcal{T})$, $\mathcal{T} = F(\mathcal{T})$.

PROOF. Suppose being minimal P is a semi-topological property. Then in view of Theorem 2.2, if $(X,\mathcal{T})$ is minimal P, $(X,F(\mathcal{T}))$ is minimal P and hence $\mathcal{T} = F(\mathcal{T})$. Conversely if $\mathcal{T} = F(\mathcal{T})$, $(X,\mathcal{T})$ is minimal P if and only if $(X,F(\mathcal{T}))$ is minimal P. Hence in view of Theorem 2.2, being minimal P is semi- topological.

We have seen that for a space $(X,\mathcal{T})$, the corresponding $F(\mathcal{T}) = \{U - N: U \in \mathcal{T}$ and N is a nowhere dense subset of $(X,\mathcal{T})\}$. So, if $(X,\mathcal{T})$ is minimal P and $(X,\mathcal{T})$ has a nonempty nowhere dense subset N such that for some $U \in \mathcal{T}$, $U - N \notin \mathcal{T}$, then for that property P, being minimal P is not a semi-topological property.

In the following example we shall use this technique to show that the property of being minimal Hausdorff, or being sequential and Hausdorff minimal or being minimal completely regular is not semi-topological.

EXAMPLE 3.16. Let $X = [0,1] \times [0,1]$ and $\mathcal{T}$ be the product topology induced by the relative usual topology on [0,1]. Let $X_n = \{\frac{1}{n}\} \times [0,1]$, $n \in N$, the set of natural numbers and let $Y = \bigcup_{n \in N} X_n$. Then Y is a nowhere dense subset of $(X,\mathcal{T})$. Also $X - Y \notin \mathcal{T}$. But being the complement of a nowhere dense subset in an open set, $X - Y \in F(\mathcal{T})$. Thus $\mathcal{T} \subsetneqq F(\mathcal{T})$.

The space $(X,\mathcal{T})$ is compact and Hausdorff and hence $(X,\mathcal{T})$ is minimal Hausdorff. But $\mathcal{T} \neq F(\mathcal{T})$. Therefore in view of Theorem 3.15 being minimal Hausdorff is not semi-topological.

The space $(X,\mathcal{T})$ is first countable and hence sequential. Also every sequential minimal Hausdorff space is sequential and Hausdorff minimal [23]. Hence $(X,\mathcal{T})$ is sequential and Hausdorff minimal. Therefore in view of Theorem 3.15 being sequential and Hausdorff minimal is not semi-topological.

The space $(X,\mathcal{T})$ is completely regular and compact and hence minimal completely regular. Thus being minimal completely regular is not semi-topological in view of Theorem 3.15.

Remark 3.17. It may be noted that there are minimal P spaces $(X,\mathcal{T})$ for which $\mathcal{T} = F(\mathcal{T})$. For, consider the space $X = \{0\} \cup N \cup \{j + \frac{1}{n}: j \geq 1, j, n \in N\}$ where N is the set of positive integers. Let $\mathcal{T}$ be the topology generated by the following basic open sets. (1) The relative basic open sets form the basic open sets in $X - \{0,1\}$. (2) All subsets of the form $\{0\} \cup \{j + \frac{1}{2n}$, $j \geq k$, $k,n \in N$ and $k \geq 2\}$, (3) All subsets of the form $\{1\} \cup \{j + \frac{1}{2n+1}$, $j \geq p$, $p,n \in N$ and $p \geq 2\}$. The nowhere dense subsets are the subsets of N and each of them is closed. Hence the complement of any nowhere dense set in an open set is open. Therefore $\mathcal{T} = F(\mathcal{T})$. It may be noted that $(X,\mathcal{T})$ is minimal Hausdorff and sequential and Hausdorff minimal.

Closely associated with the class of minimal P-spaces is the class of Katetov P spaces. A P-space $(X,\mathcal{T})$ is <u>Katetov P</u> if $\mathcal{T}$ is finer than a minimal P topology on X.

THEOREM 3.18. Suppose P is a semi-topological property and being minimal P is not semi-topological. Then there exists a minimal P-space $(X,\mathcal{T})$ for which $(X,F(\mathcal{T}))$ is Katetov P.

PROOF. In view of Theorems 2.2 and 3.15, the proof is straight forward and is omitted.

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