Internat. J. Math. & Math. Sci
VOL. 15 NO. 4 (1992) 701-718

MEASURES ON COALLOCATION AND NORMAL LATTICES

JACK-KANG CHAN

Norden Systems
75 Maxess Road
Melville, New York 11747, U.S.A.

(Received January 29, 1991 and in revised form April 15, 1991)

ABSTRACT

Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be lattices of subsets of a nonempty set X. Suppose $\mathscr{L}_2$ coallocates $\mathscr{L}_1$ and $\mathscr{L}_1$ is a subset of $\mathscr{L}_2$. We show that any $\mathscr{L}_1$-regular finitely additive measure on the algebra generated by $\mathscr{L}_1$ can be uniquely extended to an $\mathscr{L}_2$-regular measure on the algebra generated by $\mathscr{L}_2$. The case when $\mathscr{L}_1$ is not necessary contained in $\mathscr{L}_2$, as well as the measure enlargement problem are considered. Furthermore, some discussions on normal lattices and separation of lattices are also given.

KEY WORDS : lattices, normal lattices, coallocation lattices, semi-separated lattices, regular finitely additive measures, σ-smooth measures, measure extension, measure enlargement.

1980 MATHEMATICS SUBJECT CLASSIFICATION : 28C15, 28A12.

1. INTRODUCTION

Let X be an arbitrary set and $\mathscr{L}_1$ and $\mathscr{L}_2$ are lattices of subsets of X. If $\mathscr{L}_1 \subset \mathscr{L}_2$, and if $\mathscr{L}_2$ coallocates $\mathscr{L}_1$, then any $\mathscr{L}_1$-regular finitely additive measure on the algebra generated by $\mathscr{L}_1$ can be uniquely extended to an $\mathscr{L}_2$-regular measure on the algebra generated by $\mathscr{L}_2$. This situation has been investigated by J. Camacho in [2]. We extend his results in several directions in this paper. We will consider the case where $\mathscr{L}_1$ is not necessary contained in $\mathscr{L}_2$ (see Theorem 3.1) and show that under suitable conditions any $\mu \in M_R(\mathscr{L}_1)$ (see below for definitions) gives rise to a $\nu \in M_R(\mathscr{L}_2)$. We will also consider besides measure extension problems, measure enlargement problems (see e.g. Theorem 3.3) and will finally apply these results to the case of a single lattice $\mathscr{L}$, thereby extending results of M. Szeto [8] for measures on normal lattices.

We begin by giving some standard lattice and measure theoretic background in Section 2. Our notation and terminology is consistent with [1,4,6,7,9]. In Section 3, we consider the general coallocation theorem and a variety of consequences of it. Section 4 is devoted to a more detailed discussion of normal lattices and to separation of lattices. This work extends to some extent that of G. Eid [3].

2. BACKGROUND AND TERMINOLOGY

In this section, we summarize some lattice and measure theoretic notions and notations. This is all fairly standard and as previously mentioned is consistent with standard references.

Definition 2.1

Let X be a nonempty set and $\mathscr{P}(X)$ is the power set of X. A **lattice** $\mathscr{L}$ is a collection of subsets of X, which is closed under finite unions and finite intersections, and $\emptyset$, $X \in \mathscr{L}$. Let

$$\mathscr{L}' \equiv \{ L' : L \in \mathscr{L} \}$$

where L' denotes the **complement** of L. $\mathscr{L}'$ is a lattice if $\mathscr{L}$ is.

Definition 2.2

Let $\mathscr{L}$, $\mathscr{L}_1$ and $\mathscr{L}_2$ be any lattices of subsets of X.

(1) $\mathscr{L}$ is δ if it is closed under countable intersections.

(2) $\mathscr{L}$ is a **complement generated (c.g.) lattice** if

$$\forall L \in \mathscr{L},\ \exists L_1, L_2, \ldots \in \mathscr{L} \text{ such that } L = \bigcap_{n=1}^{\infty} L_n'.$$

(3) $\mathscr{L}$ is a **normal lattice** if

$$\forall L_1, L_2 \in \mathscr{L},\ L_1 \cap L_2 = \emptyset \Rightarrow$$
$$\exists \tilde{L}_1, \tilde{L}_2 \in \mathscr{L} \text{ s.t. } L_1 \subset \tilde{L}_1',\ L_2 \subset \tilde{L}_2',\ \tilde{L}_1' \cap \tilde{L}_2' = \emptyset.$$

(4) $\mathscr{L}$ is a **countably paracompact (c.p.) lattice** if

$$\forall L_1, L_2, \ldots \in \mathscr{L},\ L_1 \supset L_2 \supset \ldots,\ \lim_{n\to\infty} L_n = \emptyset\ (L_n \downarrow \emptyset) \Rightarrow$$
$$\exists \tilde{L}_1, \tilde{L}_1, \ldots \in \mathscr{L} \text{ s.t. } \forall n,\ L_n \subset \tilde{L}_n' \text{ and } \tilde{L}_n' \downarrow \emptyset.$$

(5) $\mathscr{L}_2$ is $\mathscr{L}_1$**-countably-paracompact** ($\mathscr{L}_1$**-c.p.**) if

$\forall B_1, B_2, \ldots \in \mathscr{L}_2$, $B_1 \supset B_2 \supset \ldots$, $B_n \downarrow \emptyset$ $\Rightarrow$

$\exists A_1, A_2, \ldots \in \mathscr{L}_1$ s.t. $\forall n$, $B_n \subset A_n'$ and $A_n' \downarrow \emptyset$.

(6) $\mathscr{L}_1$ **semi-separates** $\mathscr{L}_2$ if

$\forall A \in \mathscr{L}_1$, $B \in \mathscr{L}_2$, $A \cap B = \emptyset$ $\Rightarrow$ $\exists \tilde{L}_1 \in \mathscr{L}_1$, s.t. $B \subset \tilde{L}_1$ and $A \cap \tilde{L}_1 = \emptyset$.

(7) $\mathscr{L}_1$ **separates** $\mathscr{L}_2$ if

$\forall \tilde{L}_2, \hat{L}_2 \in \mathscr{L}_2$, $\tilde{L}_2 \cap \hat{L}_2 = \emptyset$ $\Rightarrow$

$\exists \tilde{L}_1, \hat{L}_1 \in \mathscr{L}_1$, s.t. $\tilde{L}_2 \subset \tilde{L}_1$, $\hat{L}_2 \subset \hat{L}_1$, and $\tilde{L}_1 \cap \hat{L}_1 = \emptyset$.

(8) $\mathscr{L}_1$ **coseparates** $\mathscr{L}_2$ if

$\forall \tilde{L}_2, \hat{L}_2 \in \mathscr{L}_2$, $\tilde{L}_2 \cap \hat{L}_2 = \emptyset$ $\Rightarrow$

$\exists \tilde{L}_1, \hat{L}_1 \in \mathscr{L}_1$, s.t. $\tilde{L}_2 \subset \tilde{L}_1'$, $\hat{L}_2 \subset \hat{L}_1'$, and $\tilde{L}_1' \cap \hat{L}_1' = \emptyset$.

(9) $\mathscr{L}_2$ **coallocates** $\mathscr{L}_1$ if

$\forall L_1 \in \mathscr{L}_1$ s.t. $L_1 \subset \tilde{L}_2' \cup \hat{L}_2'$, where $\tilde{L}_2, \hat{L}_2 \in \mathscr{L}_2$ $\Rightarrow$

$\exists \tilde{L}_1, \hat{L}_1 \in \mathscr{L}_1$ s.t. $\tilde{L}_1 \subset \tilde{L}_2'$, $\hat{L}_1 \subset \hat{L}_2'$, $L_1 = \tilde{L}_1 \cup \hat{L}_1$.

Definition 2.3

A **finitely additive (f.a.) measure** μ is a finite nonnegative function defined on the algebra $A(\mathscr{L})$ generated by $\mathscr{L}$, such that (1) $\forall A \in A(\mathscr{L})$, $\mu(A) \geq 0$, (2) $\mu(\emptyset) = 0$, and (3) [*finite additivity*] $\forall A, B \in A(\mathscr{L})$, $A \cap B = \emptyset$ $\Rightarrow$ $\mu(A \cup B) = \mu(A) + \mu(B)$.

A **0-1 measure** μ is a two-valued finitely additive measure taking value either 0 or 1.

Usually, we simply refer μ to as a *measure on a lattice* $\mathscr{L}$ to mean that μ is a finitely additive measure defined on the *algebra* $A(\mathscr{L})$.

A f.a. measure μ defined on the algebra $A(\mathscr{L})$ is

(1) **$\mathscr{L}$-regular** iff $\forall A \in A(\mathscr{L})$, $\mu(A) = \sup \{ \mu(L) : L \subset A, L \in \mathscr{L} \}$.

Or, equivalently, $\mu(A) = \inf \{ \mu(\hat{L}') : \hat{L}' \supset A, \hat{L} \in \mathscr{L} \}$.

(2) **σ-smooth on** $A(\mathscr{L})$, iff

$\forall A_1, A_2, \ldots \in A(\mathscr{L})$, $A_1 \supset A_2 \supset \ldots \downarrow \emptyset$, $(A_n \downarrow \emptyset)$ $\Rightarrow$ $\mu(A_n) \to 0$ as $n \to \infty$

(3) **σ-smooth on** $\mathscr{L}$, iff

$\forall L_1, L_2, \ldots \in \mathscr{L}$, $L_1 \supset L_2 \supset \ldots \downarrow \emptyset$, $(L_n \downarrow \emptyset)$ $\Rightarrow$ $\mu(L_n) \to 0$ as $n \to \infty$

The following notations for the collections of measures on $A(\mathscr{L})$ will be used throughout :

$M(\mathscr{L}) \equiv \{ \mu : \mu$ f.a. measure on $A(\mathscr{L}) \}$

$M_R(\mathscr{L}) \equiv \{ \mu \in M(\mathscr{L}) : \mu$ $\mathscr{L}$-regular $\}$

$M^\sigma(\mathscr{L}) \equiv \{ \mu \in M(\mathscr{L}) : \mu$ σ-smooth on $A(\mathscr{L}) \}$

$M_\sigma(\mathscr{L}) \equiv \{ \mu \in M(\mathscr{L}) : \mu$ σ-smooth on $\mathscr{L} \}$

$M_R^\sigma(\mathscr{L}) \equiv \{ \mu \in M(\mathscr{L}) : \mu$ σ-smooth on $A(\mathscr{L})$ and $\mathscr{L}$-regular $\}$

Similarly, we also define $I(\mathscr{L})$, $I_R(\mathscr{L})$, $I^\sigma(\mathscr{L})$, $I_\sigma(\mathscr{L})$, and $I_R^\sigma(\mathscr{L})$ for non-trivial 0-1 measures.

If μ is $\mathscr{L}$-regular, then σ-smoothness on $\mathscr{L}$ implies σ-smoothness on $A(\mathscr{L})$. Thus, $M_R^\sigma(\mathscr{L}) = M_R(\mathscr{L}) \cap M_\sigma(\mathscr{L}) = M_R(\mathscr{L}) \cap M^\sigma(\mathscr{L})$.
Since $A(\mathscr{L}') = A(\mathscr{L})$, we have $M(\mathscr{L}') = M(\mathscr{L})$ and $I(\mathscr{L}') = I(\mathscr{L})$.
Furthermore, μ is *σ-smooth on* $A(\mathscr{L})$ ($\mu \in M^\sigma(\mathscr{L})$) iff μ is *countably additive*.

Let $\mu_1, \mu_2 \in M(\mathscr{L})$. Define

(1) $\mu_1 \le \mu_2$ if $\forall A \in A(\mathscr{L})$, $\mu_1(A) \le \mu_2(A)$

(2) $\mu_1 \le \mu_2$ on $\mathscr{L}$, if $\forall L \in \mathscr{L}$, $\mu_1(L) \le \mu_2(L)$

(3) $\mu_1 \le \mu_2$ on $\mathscr{L}'$, if $\forall L \in \mathscr{L}$, $\mu_1(L') \le \mu_2(L')$

Definition 2.4

Suppose $\mathscr{L}_1 \subset \mathscr{L}_2$ are lattices of subsets of X such that $\mu_1 \in M(\mathscr{L}_1)$ and $\mu_2 \in M(\mathscr{L}_2)$. Denote $\mu_2|_{\mathscr{L}_1}$ (or simply $\mu_2|$) to mean the **restriction** of μ_2 to $A(\mathscr{L}_1)$.

If $\mu_1 = \mu_2|$ on $A(\mathscr{L}_1)$, then μ_2 is called a **measure extension** of μ_1 from $A(\mathscr{L}_1)$ to $A(\mathscr{L}_2)$ (or, less precisely, from $\mathscr{L}_1$ to $\mathscr{L}_2$); and a **regular measure extension**, if $\mu_2 \in M_R(\mathscr{L}_2)$.

If $\mu_1 \le \mu_2|$ on $\mathscr{L}_1$ and $\mu_1(X) = \mu_2(X)$, then μ_2 is called a **measure enlargement** of μ_1 from $A(\mathscr{L}_1)$ to $A(\mathscr{L}_2)$ (or, less precisely, from $\mathscr{L}_1$ to $\mathscr{L}_2$); and a **regular measure enlargement**, if $\mu_2 \in M_R(\mathscr{L}_2)$.

Definition 2.5

A real-valued function $\mu_* : \wp(X) \to [0,\infty)$, is called a **finitely superadditive inner measure**, if

(1) $\mu_*(\emptyset) = 0$

(2) [*nondecreasing*] $\forall A \subset B \subset X \Rightarrow \mu_*(A) \le \mu_*(B)$, that is, $\mu_* \uparrow$

(3) [*finite superadditivity*] $\forall A,B \subset X$, $A \cap B = \emptyset \Rightarrow \mu_*(A \cup B) \ge \mu_*(A) + \mu_*(B)$

A real-valued function $\mu^* : \wp(X) \to [0,\infty)$, is called a **finitely subadditive outer measure**, if it satisfies (1), (2) and

(3') [*finite subadditivity*] $\forall A,B \subset X$, $A \cap B = \emptyset \Rightarrow \mu^*(A \cup B) \le \mu^*(A) + \mu^*(B)$

Let μ^* be a finitely subadditive outer measure on $(X,\mathscr{L})$. A set $E \subset X$ is said to be **μ^*-measurable**, if

$$\mu^*(T) = \mu^*(T \cap E) + \mu^*(T \cap E'), \quad \forall T \subset X .$$

We have the following theorem characterizing a normal lattice as a special case of the *coallocation property* :

THEOREM 2.1

$\mathscr{L}$ is *normal* $\Leftrightarrow$ $\forall L\in\mathscr{L}$ s.t. $L\subset L_1'\cup L_2'$, where $L_1, L_2\in\mathscr{L}$ $\Rightarrow$ $\exists \tilde{L}_1, \tilde{L}_2\in\mathscr{L}$ s.t. $L_1\subset\tilde{L}_1'$, $L_2\subset\tilde{L}_2'$, $L=\tilde{L}_1\cup\tilde{L}_2$.

Proof:

"$\Leftarrow$"

Suppose $L_1, L_2\in\mathscr{L}$ and $L_1\cap L_2=\emptyset$. $\therefore X=L_1'\cup L_2'$. Then by assumption, $\exists \tilde{L}_1, \tilde{L}_2\in\mathscr{L}$ such that $\tilde{L}_1\subset L_1', \tilde{L}_2\subset L_2'$ and $X=\tilde{L}_1\cup\tilde{L}_2$ or $\tilde{L}_1'\cap\tilde{L}_2'=\emptyset$.
Thus, when $L_1\cap L_2=\emptyset$, $L_1\subset\tilde{L}_1'$, $L_2\subset\tilde{L}_2'$, we have $\tilde{L}_1'\cap\tilde{L}_2'=\emptyset$. $\therefore\mathscr{L}$ is *normal*.

"$\Rightarrow$"

Let $L\in\mathscr{L}$ and $L\subset L_1'\cup L_2'$ where $L_1, L_2\in\mathscr{L}$. Consider $L-L_1'$ and $L-L_2'$, $(L-L_1')\cap(L-L_2') = (L\cap L_1)\cap(L\cap L_2) = L\cap(L_1'\cup L_2')' = \emptyset$. $\therefore L-L_1'$ and $L-L_2'$ are disjoint. By normality, $\exists \hat{L}_1, \hat{L}_2\in\mathscr{L}$, such that $(L-L_1')\subset\hat{L}_1'$, $(L-L_2')\subset\hat{L}_2'$ and $\hat{L}_1'\cap\hat{L}_2'=\emptyset$, or $\hat{L}_1\cup\hat{L}_2=X$. Define

$$\tilde{L}_1\equiv(L-\hat{L}_1') = L\cap\hat{L}_1\in\mathscr{L} \quad\text{and}\quad \tilde{L}_2\equiv(L-\hat{L}_2') = L\cap\hat{L}_2\in\mathscr{L}$$

Then $\tilde{L}_1\cup\tilde{L}_2 = (L\cap\hat{L}_1)\cup(L\cap\hat{L}_2) = L\cap(\hat{L}_1\cup\hat{L}_2) = L\cap X = L$. Now,

$$L-L_1'\subset\hat{L}_1' \Rightarrow \hat{L}_1\subset L'\cup L_1' \quad\text{and}\quad L-L_2'\subset\hat{L}_2' \Rightarrow \hat{L}_2\subset L'\cup L_2'$$

$\therefore$

$$\tilde{L}_1 = L\cap\hat{L}_1\subset L\cap(L'\cup L_1') = (L\cap L')\cup(L\cap L_1') = (L\cap L_1')\subset L_1'$$

and

$$\tilde{L}_2 = L\cap\hat{L}_2\subset L\cap(L'\cup L_2') = (L\cap L')\cup(L\cap L_2') = (L\cap L_2')\subset L_2'$$

Thus, $\forall L\in\mathscr{L}$, $L\subset L_1'\cup L_2'$, $\exists \tilde{L}_1,\tilde{L}_2\in\mathscr{L}$, s.t. $\tilde{L}_1\subset L_1'$, $\tilde{L}_2\subset L_2'$ and $L=\tilde{L}_1\cup\tilde{L}_2$. ∎

The following results are obvious :

(1) $\mathscr{L}$ *is normal* $\Leftrightarrow$ $\mathscr{L}$ *coallocates itself* $\Leftrightarrow$ $\mathscr{L}$ *coseparates itself*.

(2) $\mathscr{L}_1$ *separates* $\mathscr{L}_2$ $\Rightarrow$ $\mathscr{L}_1$ *semi-separates* $\mathscr{L}_2$.

Furthermore, we have the following measure theoretic characterization of a normal lattice :

THEOREM 2.2

$\mathscr{L}$ is *normal* iff

$\forall \mu\in I(\mathscr{L})$, s.t. on $\mathscr{L}$, $\mu\le\nu_1\in I_R(\mathscr{L})$, $\mu\le\nu_2\in I_R(\mathscr{L})$ $\Rightarrow$ $\nu_1=\nu_2$.

THEOREM 2.3

Suppose $\mathscr{L}_1\subset\mathscr{L}_2$. Then $\mathscr{L}_1$ *coseparates* $\mathscr{L}_2$ $\Rightarrow$ $\mathscr{L}_2$ *coallocates* $\mathscr{L}_1$.

Proof:

Suppose $L_1\subset\tilde{L}_2'\cup\hat{L}_2'$, where $L_1\in\mathscr{L}_1$, $\tilde{L}_2, \hat{L}_2\in\mathscr{L}$. Then,
$L_1\cap\tilde{L}_2, L_1\cap\hat{L}_2\in\mathscr{L}_2\supset\mathscr{L}_1$.

Now $(L_1\cap\tilde{L}_2)\cap(L_1\cap\hat{L}_2) = L_1\cap(\tilde{L}_2\cap\hat{L}_2) = L_1\cap(\tilde{L}_2'\cup\hat{L}_2')' = \emptyset$.

$\mathscr{L}_1$ coseparates $\mathscr{L}_2\Rightarrow\exists\tilde{L}_1,\hat{L}_1\in\mathscr{L}_1$ s.t. $\tilde{L}_1'\cap\hat{L}_1'=\emptyset$, and

$L_1 \cap \tilde{L}_2 \subset \tilde{L}_1'$ and $L_1 \cap \hat{L}_2 \subset \hat{L}_1'$.

Define $\tilde{L}_1^\circ \equiv L_1 \cap \tilde{L}_1$ and $\hat{L}_1^\circ \equiv L_1 \cap \hat{L}_1$, hence $\tilde{L}_1^\circ, \hat{L}_1^\circ \in \mathscr{L}_1$. And

$$\tilde{L}_1^\circ \cup \hat{L}_1^\circ = (L_1 \cap \tilde{L}_1) \cup (L_1 \cap \hat{L}_1) = L_1 \cap (\tilde{L}_1 \cup \hat{L}_1) = L_1 \cap (\tilde{L}_1' \cap \hat{L}_1')' = L_1.$$

$\therefore L_1 = \tilde{L}_1^\circ \cup \hat{L}_1^\circ$. Now

$\tilde{L}_1^\circ = L_1 \cap \tilde{L}_1 \subset L_1 \cap (L_1 \cap \tilde{L}_2)' = L_1 \cap (L_1' \cup \tilde{L}_2') = (L_1 \cap L_1') \cup (L_1 \cap \tilde{L}_2')$

$= L_1 \cap \tilde{L}_2' \subset \tilde{L}_2'$.

Thus, $\tilde{L}_1^\circ \subset \tilde{L}_2'$. Similarly, $\hat{L}_1^\circ \subset \hat{L}_2'$. Hence $\mathscr{L}_2$ coallocates $\mathscr{L}_1$. ∎

THEOREM 2.4

$\mathscr{L}$ *countably paracompact* $\Rightarrow M_\sigma(\mathscr{L}') \subset M_\sigma(\mathscr{L})$

Proof:

Suppose $\forall n$, $L_n \in \mathscr{L}$, $L_n \downarrow \emptyset$

$\mathscr{L}$ c.p. $\Rightarrow \exists \tilde{L}_n \in \mathscr{L}$, s.t. $L_n \subset \tilde{L}_n'$, $\tilde{L}_n' \downarrow \emptyset$

$\mu \in M_\sigma(\mathscr{L}') \Leftrightarrow \mu(\tilde{L}_n') \to 0$ as $\tilde{L}_n' \downarrow \emptyset$

$\therefore \mu(L_n) \le \mu(\tilde{L}_n') \to 0$, $\Rightarrow \mu(L_n) \to 0$

Now $L_n \downarrow \emptyset$, $\mu(L_n) \to 0$ $\therefore \mu \in M_\sigma(\mathscr{L})$. Hence, $M_\sigma(\mathscr{L}') \subset M_\sigma(\mathscr{L})$. ∎

3. MEASURES ON COALLOCATION LATTICES

In this section we extend some of the work of [8] and [2] on the unique extendability of a measure $\mu \in M_R(\mathscr{L}_1)$ to a measure $\nu \in M_R(\mathscr{L}_2)$ where $\mathscr{L}_1$ and $\mathscr{L}_2$ are lattices of subsets of X. We note that it is not always necessary to assume that $\mathscr{L}_1 \subset \mathscr{L}_2$ nor that X belongs to the lattices in order for the main results of the **coallocation theorem** to hold (see Theorem 3.1). We first define two functions which form an inner-outer measure pair.

Definition 3.1

Suppose $\mathscr{L}_1$ and $\mathscr{L}_2$ are lattices of subsets of X and $\mu \in M(\mathscr{L}_1)$. For all $E \subset X$, define

$$\mu_\bullet(E) \equiv \sup \{ \mu(L_1) : E \supset L_1,\ L_1 \in \mathscr{L}_1 \}$$

and

$$\mu^\wedge(E) \equiv \inf \{ \mu_\bullet(L_2') : E \subset L_2',\ L_2 \in \mathscr{L}_2 \}$$

We have the following :

THEOREM 3.1 [*Coallocation theorem*]

Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be lattices of subsets of $X \neq \emptyset$. Suppose $\mu \in M(\mathscr{L}_1)$. We have

(1) $\mu_\bullet$ is a **finitely superadditive inner measure**

(2) $\mathcal{L}_2$ *coallocates* $\mathcal{L}_1$ $\Rightarrow$ $\mu_\bullet$ is finitely additive on $\mathcal{L}_2'$

(3) $\mathcal{L}_2$ *coallocates* $\mathcal{L}_1$ $\Rightarrow$ $\mu\hat{}$ is a **finitely subadditive outer measure**

(4) $\mu\hat{} = \mu_\bullet$ *on* $\mathcal{L}_2'$

In particular, if $X \in \mathcal{L}_1$ *and* $\emptyset \in \mathcal{L}_2$, then $\mu\hat{}(X) = \mu_\bullet(X) = \mu(X)$

(5) [a] $\mu \le \mu\hat{}$ *on* $\mathcal{L}_1$

[b] $\mathcal{L}_1 \subset \mathcal{L}_2$ *and* $\mu \in M_R(\mathcal{L}_1)$ $\Rightarrow$ $\mu_\bullet = \mu$ *on* $\mathcal{L}_1'$; $\mu\hat{} = \mu$ *on* $\mathcal{L}_1$

(6) Suppose $\mathcal{L}_2$ *coallocates* $\mathcal{L}_1$. $E \subset X$ is **$\mu\hat{}$-measurable**

$\Leftrightarrow$ $\forall L_2 \in \mathcal{L}_2$, $\mu\hat{}(L_2') \ge \mu\hat{}(L_2' \cap E) + \mu\hat{}(L_2' \cap E')$

(7) Suppose $\mathcal{L}_2$ *coallocates* $\mathcal{L}_1$.

If either [a] $\mathcal{L}_1 \subset \mathcal{L}_2$

or [b] $\mathcal{L}_2$ *semi-separates* $\mathcal{L}_1$

then

[1°] every element of $\mathcal{L}_2'$ is $\mu\hat{}$*-measurable*

[2°] $\mu\hat{}|_{\mathcal{L}_2}$ is a *finitely additive measure on* $A(\mathcal{L}_2)$

[3°] $\mu\hat{}$ is $\mathcal{L}_1$*-regular on* $\mathcal{L}_2'$

[4°] $\mu\hat{} \in M_R(\mathcal{L}_2)$.

Proof:

(1) The proof is standard and is therefore omitted.

(2) Let $A_2, B_2 \in \mathcal{L}_2$, and $L_1 \in \mathcal{L}$ s.t. $L_1 \subset A_2' \cup B_2' \in \mathcal{L}_2'$

$\mathcal{L}_2$ coallocates $\mathcal{L}_1$ $\Rightarrow$

$\exists\ A_1, B_1 \in \mathcal{L}_1$ s.t. $A_1 \subset A_2'$, $B_1 \subset B_2'$, and $L_1 = A_1 \cup B_1$

$$\begin{aligned}\therefore\ \mu(L_1) &= \mu(A_1 \cup B_1)\\ &\le \mu(A_1) + \mu(B_1)\\ &\le \sup\{\ \mu(A_1)\ :\ A_2' \supset A_1\ \} + \sup\{\ \mu(B_1)\ :\ B_2' \supset B_1\ \}\\ &\equiv \mu_\bullet(A_2') + \mu_\bullet(B_2')\end{aligned}$$

Taking sup on the left hand side,

$$\sup\{\ \mu(L_1)\ :\ L_1 \subset A_2' \cup B_2'\ \} \le \mu_\bullet(A_2') + \mu_\bullet(B_2')$$

$\therefore\ \mu_\bullet(A_2' \cup B_2') \le \mu_\bullet(A_2') + \mu_\bullet(B_2')$ $\Rightarrow$

$\mu_\bullet$ is finitely subadditive on $\mathcal{L}_2'$. Together with (1), $\mu_\bullet$ is finitely additive on $\mathcal{L}_2'$.

(3) The proof is also standard and is omitted.

(4) Let $L_2 \in \mathcal{L}_2$.

$\mu\hat{}(L_2') = \inf\ \{\ \mu_\bullet(\tilde{L}_2')\ :\ L_2' \subset \tilde{L}_2',\ \tilde{L}_2 \in \mathcal{L}_2\ \}$

$\Rightarrow$ $\mu\hat{}(L_2') \le \mu_\bullet(L_2')$ ....[i]

Now if $A_2' \in \mathscr{L}_2'$ and $L_2' \subset A_2'$, then by monotonicity of $\mu_\cdot$,

$\mu_\cdot(L_2') \le \mu_\cdot(A_2')$

$\Rightarrow \mu_\cdot(L_2') \le \inf\{\mu_\cdot(A_2') : L_2' \subset A_2',\ A_2 \in \mathscr{L}_2\} = \mu\hat{}(L_2')$ ......[ii]

[i] and [ii] $\Rightarrow$ $\mu\hat{} = \mu_\cdot$ on $\mathscr{L}_2'$.

If $X \in \mathscr{L}_1$, take $L_1 = X$, $\mu_\cdot(X) = \sup\{\mu(L_1) : X \supset L_1 \in \mathscr{L}_1\} = \mu(X)$.

If $\emptyset \in \mathscr{L}_2$, $X = \emptyset' \in \mathscr{L}_2'$ and $\mu\hat{} = \mu_\cdot$ on $\mathscr{L}_2' \Rightarrow \mu\hat{}(X) = \mu_\cdot(X)$

Consequently, $\mu\hat{}(X) = \mu_\cdot(X) = \mu(X)$.

(5)[a] Let $L_1 \in \mathscr{L}_1$ and $A_2 \in \mathscr{L}_2$, s.t. $L_1 \subset A_2'$

$\mu_\cdot(A_2') = \sup\{\mu(L_1) : A_2' \supset L_1 \in \mathscr{L}_1\} \ge \mu(L_1)$

Taking inf, $\inf\{\mu_\cdot(A_2') : L_1 \subset A_2',\ A_2 \in \mathscr{L}_2\} \ge \mu(L_1)$

i.e. $\mu\hat{}(L_1) \ge \mu(L_1)$. Or, $\mu \le \mu\hat{}$ on $\mathscr{L}_1$.

(5)[b] Suppose $\mathscr{L}_1 \subset \mathscr{L}_2$, then $\mu\hat{} = \mu_\cdot$ on $\mathscr{L}_2' \Rightarrow \mu\hat{} = \mu_\cdot$ on $\mathscr{L}_1'$

$\therefore$ if $\tilde{L}_1 \in \mathscr{L}_1$, then $\mu\hat{}(\tilde{L}_1') = \mu_\cdot(\tilde{L}_1')$

Suppose $L_1 \subset A_2' \subset A_1'$, where $A_1 \in \mathscr{L}_1$, $A_2 \in \mathscr{L}_2$, $A_1 \subset A_2$

$\mu_\cdot(A_1') = \sup\{\mu(\tilde{L}_1) : A_1' \subset \tilde{L}_1 \in \mathscr{L}_1\}$

But $\mu \in M_R(\mathscr{L}_1) \Rightarrow \mu(A_1') = \sup\{\mu(\tilde{L}_1) : \tilde{L}_1 \in \mathscr{L}_1\}$ whenever $A_1' \subset \tilde{L}_1$

$\therefore \mu_\cdot(A_1') = \mu(A_1')$ $\forall A_1' \in \mathscr{L}_1'$, hence $\mu_\cdot = \mu$ on $\mathscr{L}_1'$.

Now, $\mu\hat{}(L_1) = \inf\{\mu_\cdot(A_2') : L_1 \subset A_2',\ A_2 \in \mathscr{L}_2\}$

$\le \inf\{\mu_\cdot(A_1') : L_1 \subset A_1',\ A_1 \in \mathscr{L}_1\}$ $(\because A_2' \subset A_1')$

$= \inf\{\mu(A_1') : L_1 \subset A_1',\ A_1 \in \mathscr{L}_1\}$ $(\because \mu_\cdot = \mu$ on $\mathscr{L}_1')$

$= \mu(L_1)$ $(\because \mu \in M_R(\mathscr{L}_1))$

$\therefore \mu\hat{} \le \mu$ on $\mathscr{L}_1$, and (5)[a] $\Rightarrow$ $\mu \le \mu\hat{}$ on $\mathscr{L}_1$, $\therefore$ $\mu\hat{} = \mu$ on $\mathscr{L}_1$.

(6) "$\Leftarrow$" Suppose $\forall A_2 \in \mathscr{L}_2$, we have $\forall E \subset X$,

$\mu\hat{}(A_2') \ge \mu\hat{}(A_2' \cap E) + \mu\hat{}(A_2' \cap E')$

Suppose $T \subset X$, s.t. $T \subset A_2'$, $A_2 \in \mathscr{L}_2$

$\mu\hat{}(T) = \inf\{\mu_\cdot(A_2') : T \subset A_2',\ A_2 \in \mathscr{L}_2\}$

Now (4) $\Rightarrow$

$\mu_\cdot(A_2') = \mu\hat{}(A_2')$

$\ge \mu\hat{}(A_2' \cap E) + \mu\hat{}(A_2' \cap E')$ (by assumption)

$\ge \mu\hat{}(T \cap E) + \mu\hat{}(T \cap E')$ $(\because T \subset A_2',\ \mu\hat{}\ \uparrow)$

Taking inf,

$\mu\hat{}(T) \ge \mu\hat{}(T \cap E) + \mu\hat{}(T \cap E')$ .... [iii]

$\mu\hat{}$ is finitely subadditive $\Rightarrow$

$\mu\hat{}(T) = \mu\hat{}(T \cap (E \cup E')) \le \mu\hat{}(T \cap E) + \mu\hat{}(T \cap E')$ .... [iv]

[iii] and [iv] $\Rightarrow$ $\quad \mu^{\wedge}(T) = \mu^{\wedge}(T\cap E) + \mu^{\wedge}(T\cap E') \quad \forall\, T \subset X$,

which is the definition of E to be $\mu^{\wedge}$-measurable.

(6) "$\Rightarrow$" By the definition of E to be $\mu^{\wedge}$-measurable, we have

$$\mu^{\wedge}(T) = \mu^{\wedge}(T\cap E) + \mu^{\wedge}(T\cap E') \quad \forall\, T \subset X,$$

But $\mu^{\wedge}$ is finitely subadditive, the above is equivalent to

$$\mu^{\wedge}(T) \geq \mu^{\wedge}(T\cap E) + \mu^{\wedge}(T\cap E') \quad \forall\, T \subset X,$$

In particular, take $T = L_2' \in \mathscr{L}_2'$, we have

$$\mu^{\wedge}(L_2') \geq \mu^{\wedge}(L_2'\cap E) + \mu^{\wedge}(L_2'\cap E') \quad \forall\, L_2 \in \mathscr{L}_2 .$$

(7) Suppose $\mathscr{L}_2$ coallocates $\mathscr{L}_1$.

Let $L_2' \in \mathscr{L}_2'$. To prove that L_2' is $\mu^{\wedge}$-measurable, we have to show, by (6), that

$$\mu^{\wedge}(A_2') \geq \mu^{\wedge}(A_2'\cap L_2') + \mu^{\wedge}(A_2'\cap L_2) \quad \forall\, A_2 \in \mathscr{L}_2 .$$

$\forall\, A_2 \in \mathscr{L}_2$, let $P, Q \in \mathscr{L}_1$ ($\therefore\ P\cup Q \in \mathscr{L}_1$) s.t.

$$P \subset A_2' \cap L_2' \quad \text{and} \quad Q \subset A_2' \cap P'$$

Thus, $\quad P \subset A_2' \quad$ and $\quad Q \subset A_2'$

Now, $\quad P \cup Q \subset (A_2' \cap L_2') \cup (A_2' \cap P') \subset A_2'$

and $\quad P \cap Q \subset P \cap (A_2' \cap P') = \emptyset$

$$\begin{aligned}
\mu^{\wedge}(A_2') &= \mu_{*}(A_2') \qquad (\mu^{\wedge} = \mu_{*} \text{ on } \mathscr{L}_2') \\
&\geq \sup\{\, \mu(P\cup Q) : A_2' \supset P\cup Q \in \mathscr{L}_1 \,\} \\
&\geq \mu(P\cup Q) \\
&= \mu(P) + \mu(Q) \qquad (\, P\cap Q = \emptyset \,)
\end{aligned}$$

$\Rightarrow$
$$\begin{aligned}
\mu^{\wedge}(A_2') &\geq \mu(P) + \sup\{\, \mu(Q) : A_2' \cap P' \supset Q \in \mathscr{L}_1 \,\} \\
&= \mu(P) + \mu_{*}(A_2' \cap P')
\end{aligned}$$

$\Rightarrow$
$$\mu^{\wedge}(A_2') \geq \mu(P) + \mu_{*}(A_2' \cap P') \quad \text{....................[v]}$$

(7)[a] : Suppose $\mathscr{L}_1 \subset \mathscr{L}_2$.

If $\mathscr{L}_1 \subset \mathscr{L}_2$, then $P \in \mathscr{L}_1 \Rightarrow P \in \mathscr{L}_2 \quad \therefore\ A_2 \cup P \in \mathscr{L}_2$

$\Rightarrow$ $A_2' \cap P' = (A_2 \cup P)' \in \mathscr{L}_2'$, and $\mu^{\wedge} = \mu_{*}$ on $\mathscr{L}_2'$, $\therefore$ [v] $\Rightarrow$

$$\begin{aligned}
\mu^{\wedge}(A_2') &\geq \mu(P) + \mu^{\wedge}(A_2' \cap P') \\
&\geq \mu(P) + \mu^{\wedge}(A_2' \cap L_2) \qquad (\, P \subset L_2' \,)
\end{aligned}$$

$\Rightarrow$
$$\begin{aligned}
\mu^{\wedge}(A_2') &\geq \sup\{\, \mu(P) : A_2' \cap L_2' \supset P \in \mathscr{L}_1 \,\} + \mu^{\wedge}(A_2' \cap L_2) \\
&\equiv \mu_{*}(A_2' \cap L_2') + \mu^{\wedge}(A_2' \cap L_2) \\
&= \mu^{\wedge}(A_2' \cap L_2') + \mu^{\wedge}(A_2' \cap L_2) \quad \forall\, A_2 \in \mathscr{L}_2 \ (\, \mu^{\wedge} = \mu_{*} \text{ on } \mathscr{L}_2' \,)
\end{aligned}$$

We conclude, from (6), that every element of $\mathscr{L}_2'$ is $\mu^{\wedge}$-measurable.

$\therefore\ A(\mathscr{L}_2) = A(\mathscr{L}_2') \subset \{\, \mu^{\wedge}\text{-measurable sets} \,\}$. By a standard Carathéodory argument, $\mu^{\wedge}|_{\mathscr{L}_2}$ is a *finitely additive measure on* $A(\mathscr{L}_2)$.

Suppose $L_2 \in \mathscr{L}_2$,

$$\begin{aligned} \mu^{\wedge}(L_2') &= \mu_*(L_2') && \text{(by (4))} \\ &= \sup\{\ \mu(L_1)\ :\ L_1 \subset L_2',\ L_1 \in \mathscr{L}_1\ \} \\ &\le \sup\{\ \mu^{\wedge}(L_1)\ :\ L_1 \subset L_2',\ L_1 \in \mathscr{L}_1\ \} && \text{(by (5)[a])} \end{aligned}$$

But $L_1 \subset L_2' \Rightarrow \mu^{\wedge}(L_1) \le \mu^{\wedge}(L_2')$, taking sup $\Rightarrow$

$$\sup\{\ \mu^{\wedge}(L_1)\ :\ L_1 \subset L_2',\ L_1 \in \mathscr{L}_1\ \} \le \mu^{\wedge}(L_2')$$

Hence,

$$\mu^{\wedge}(L_2') = \sup\{\ \mu^{\wedge}(L_1)\ :\ L_1 \subset L_2',\ L_1 \in \mathscr{L}_1\ \}$$

which means that $\mu^{\wedge}$ is $\mathscr{L}_1$*-regular on* $\mathscr{L}_2'$. Since $\mathscr{L}_1 \subset \mathscr{L}_2$, $\mu^{\wedge}$ is also $\mathscr{L}_2$*-regular on* $\mathscr{L}_2'$. Now any element of $A(\mathscr{L}_2)$ is, of the form

$$\bigcup_{i=1}^{n} (A_i \cap B_i') \qquad A_i, B_i \in \mathscr{L}_2$$

Consequently, $\mu^{\wedge} \in M_R(\mathscr{L}_2)$.

(7)[b] : Suppose $\mathscr{L}_2$ *semi-separates* $\mathscr{L}_1$.

Now $P \subset A_2' \cap L_2' \Rightarrow P \subset L_2' \Rightarrow P \cap L_2 = \emptyset$

and $P \in \mathscr{L}_1$, $L_2 \in \mathscr{L}_2$.

$\mathscr{L}_2$ *semi-separates* $\mathscr{L}_1$ $\Rightarrow$

$\exists \tilde{L}_2 \in \mathscr{L}_2$ s.t. $P \subset \tilde{L}_2 \subset L_2'$. $\therefore P' \supset \tilde{L}_2' \supset L_2 \Rightarrow A_2' \cap P' \supset A_2' \cap \tilde{L}_2'$

From [v],

$$\begin{aligned} \mu^{\wedge}(A_2') &\ge \mu(P) + \mu_*(A_2' \cap P') \\ &\ge \mu(P) + \mu_*(A_2' \cap \tilde{L}_2') \\ &= \mu(P) + \mu^{\wedge}(A_2' \cap \tilde{L}_2') && \text{(by (4))} \\ &\ge \mu(P) + \mu^{\wedge}(A_2' \cap L_2) && (\ \tilde{L}_2' \supset L_2\ ) \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} \mu^{\wedge}(A_2') &\ge \sup\{\ \mu(P)\ :\ A_2' \cap L_2' \supset P \in \mathscr{L}_1\ \} + \mu^{\wedge}(A_2' \cap L_2) \\ &= \mu_*(A_2' \cap L_2') + \mu^{\wedge}(A_2' \cap L_2) \\ &= \mu^{\wedge}(A_2' \cap L_2') + \mu^{\wedge}(A_2' \cap L_2), \quad \forall A_2 \in \mathscr{L}_2\ (\ \mu^{\wedge} = \mu_* \text{ on } \mathscr{L}_2'\ ) \end{aligned}$$

We conclude, from (6), that every element of $\mathscr{L}_2'$ is $\mu^{\wedge}$-measurable.

$\therefore A(\mathscr{L}_2) = A(\mathscr{L}_2') \subset \{\ \mu^{\wedge}\text{-measurable sets}\ \}$. By a standard Carathéodory argument, $\mu^{\wedge}|_{\mathscr{L}_2}$ is a *finitely additive measure on* $A(\mathscr{L}_2)$.

Let $L_2 \in \mathscr{L}_2$. Suppose $L_1 \subset L_2'$ $L_1 \in \mathscr{L}_1$. $\mathscr{L}_2$ *semi-separates* $\mathscr{L}_1$

$\Rightarrow$ $\exists \tilde{L}_2 \in \mathscr{L}_2$ s.t. $L_1 \subset \tilde{L}_2 \subset L_2'$ and $\mu^{\wedge} = \mu$ on $\mathscr{L}_1$,

$$\begin{aligned} \mu(L_1) &= \mu^{\wedge}(L_1) \\ &\le \mu^{\wedge}(\tilde{L}_2) \le \mu^{\wedge}(L_2') \end{aligned}$$

$\therefore$

$$\mu(L_1) \le \mu^{\wedge}(\tilde{L}_2) \le \mu^{\wedge}(L_2')$$

Taking sup,

$$\sup\{\ \mu(L_1) : L_1 \subset L_2',\ L_1 \in \mathscr{L}_1\ \} \le \sup\{\ \mu^{\wedge}(\tilde{L}_2)\ :\ \tilde{L}_2 \subset L_2',\ \tilde{L}_2 \in \mathscr{L}_2\ \} \le \mu^{\wedge}(L_2')$$

But $\mu\hat{}(L_2') = \sup\{ \mu(L_1) . L_1 \subset L_2', L_1 \in \mathscr{L}_1 \}$ Hence,

$\mu\hat{}(L_2') = \sup\{ \mu(L_1) : L_1 \subset \hat{L}_2 , L_1 \in \mathscr{L}_1 \} = \sup\{ \mu\hat{}(\hat{L}_2) : \hat{L}_2 \subset L_2', \hat{L}_2 \in \mathscr{L}_2 \}$

$\therefore$ $\mu\hat{}$ is $\mathscr{L}_1$*-regular on* $\mathscr{L}_2'$ and $\mathscr{L}_2$*-regular on* $\mathscr{L}_2'$, and consequently, $\mu\hat{} \in M_R(\mathscr{L}_2)$.

Note : If $\mathscr{L}_1 \subset \mathscr{L}_2$, then $\mathscr{L}_2$ trivially semi-separates $\mathscr{L}_1$, (7)[a] $\Rightarrow$ (7)[b]. ∎

Corollary 3.1

Suppose $\mathscr{L}_1 = \mathscr{L}_2 = \mathscr{L}$, $X \in \mathscr{L}$, and $\mathscr{L}$ *coallocates itself*, ($\mathscr{L}$ is *normal*).

then

(1) $\mu\hat{}$ is *finitely additive* and $\mu\hat{}(X) = \mu_{\bullet}(X) = \mu(X)$

(2) $\mu\hat{}(L) + \mu_{\bullet}(L') = \mu(X)$ $\forall L \in \mathscr{L}$

Proof:

(1) Direct consequences of Theorem 3.1.

(2) From Theorem 3.1(4), $\mu\hat{} = \mu_{\bullet}$ on $\mathscr{L}_2' = \mathscr{L}'$. $\therefore$ $\mu\hat{}(L') = \mu_{\bullet}(L')$ $\forall L \in \mathscr{L}$

Now $\mu\hat{}$ is finitely additive,

$$\mu\hat{}(X) = \mu\hat{}(L \cup L') = \mu\hat{}(L) + \mu\hat{}(L') = \mu\hat{}(L) + \mu_{\bullet}(L')$$

But $\mu\hat{}(X) = \mu_{\bullet}(X) = \mu(X)$, $\therefore$ $\mu\hat{}(L) + \mu_{\bullet}(L') = \mu(X)$. ∎

The coallocation theorem leads to the following direct consequences whose proofs are omitted.

THEOREM 3.2 [*Regular measure extension on coallocation lattices*]

Suppose $\mathscr{L}_1 \subset \mathscr{L}_2$ and $\mu \in M_R(\mathscr{L}_1)$. If $\mathscr{L}_2$ *coallocates* $\mathscr{L}_1$, then there *exists* a *unique* $\nu \in M_R(\mathscr{L}_2)$, s.t. on $\mathscr{L}_1$, $\mu = \nu|_{\mathscr{L}_1} \in M_R(\mathscr{L}_1)$.

Furthermore, ν is $\mathscr{L}_1$*-regular on* $\mathscr{L}_2'$. Note that $\nu = \hat{\mu}\hat{}|_{\mathscr{L}_2}$.

THEOREM 3.3 [*Regular measure enlargement on coallocation lattices*]

Suppose $\mathscr{L}_1 \subset \mathscr{L}_2$ and $\mu \in M(\mathscr{L}_1)$. If $\mathscr{L}_2$ *coallocates* $\mathscr{L}_1$, then $\exists\ \nu \in M_R(\mathscr{L}_2)$, s.t. $\mu \le \nu$ on $\mathscr{L}_1$ and $\mu(X) = \nu(X)$.

THEOREM 3.4 [*Regular measure enlargement on a normal lattice*]

Suppose $\mathscr{L}$ is *normal* and $\mu \in M(\mathscr{L})$. Then there exists a unique $\nu \in M_R(\mathscr{L})$, s.t. $\mu \le \nu$ on $\mathscr{L}$ and $\mu(X) = \nu(X)$.

Furthermore, if we impose a σ-smoothness condition on μ, we obtain the following :

THEOREM 3.5

Suppose $\mathscr{L}_1 \subset \mathscr{L}_2$, and $\mathscr{L}_2$ *coallocates* $\mathscr{L}_1$, and $\mu \in M_R^{\sigma}(\mathscr{L}_1)$,

$\nu \in M_R(\mathscr{L}_2)$, where ν is the regular measure extension of μ. Then

$$\nu \in M_R(\mathscr{L}_2) \cap M_\sigma(\mathscr{L}_2').$$

Proof:

$\mu \in M_R^\sigma(\mathscr{L}_1) \Rightarrow \mu \in M_R(\mathscr{L}_1)$ } hence by Theorem 3.2,

$\nu \equiv \mu^\wedge|_{\mathscr{L}_2} \in M_R(\mathscr{L}_2)$ is the unique regular measure extension of μ.

Theorem 3.1(7) $\Rightarrow \mu^\wedge$ is $\mathscr{L}_1$-regular on $\mathscr{L}_2'$.

$\Rightarrow \forall\ B_n' \in \mathscr{L}_2',\ B_n' \downarrow \emptyset,\ \exists\ A_n \in \mathscr{L}_1,\ A_n \subset B_n',\ \forall \varepsilon > 0,$

$\mu^\wedge(B_n') < \mu^\wedge(A_n) + \varepsilon/2$

$\mu^\wedge = \mu$ on $\mathscr{L}_1$, since μ is $\mathscr{L}_1$-regular [Theorem 3.1(5)[b]]

$\mu^\wedge(B_n') < \mu(A_n) + \varepsilon/2$ .... [i]

Since $A_n \subset B_n' \downarrow \emptyset$, without the loss of generality, we may assume $A_n \downarrow \emptyset$. Thus, $\mu(A_n) \to 0$ $\because$ μ is σ-smooth on $\mathscr{L}_1$

Hence, [i] $\Rightarrow$ $\mu^\wedge(B_n') < \varepsilon$

Consequently, $\mu^\wedge(B_n') \to 0 \quad \forall\ B_n' \in \mathscr{L}_2'$

Or, $\nu = \mu^\wedge|_{\mathscr{L}_2} \in M_R(\mathscr{L}_2) \cap M_\sigma(\mathscr{L}_2')$.

∎

In particular, if $\mathscr{L}_1 = \mathscr{L}_2 = \mathscr{L}$ is normal, we have

Corollary 3.5

Suppose $\mathscr{L}$ is *normal*, and $\mu \in M_\sigma(\mathscr{L})$, $\nu \in M_R(\mathscr{L})$, ν is the regular measure enlargement of μ, $\mu \le \nu$ on $\mathscr{L}$, $\mu(X) = \nu(X)$. Then, $\nu \in M_\sigma(\mathscr{L}')$.

Theorem 3.6

Suppose $\mathscr{L}_1 \subset \mathscr{L}_2$, and $\mathscr{L}_2$ *coallocates* $\mathscr{L}_1$, and $\mathscr{L}_2$ is *countably paracompact* and *normal*. Suppose $\mu \in M_R^\sigma(\mathscr{L}_1)$, and $\nu \in M_R(\mathscr{L}_2)$, where ν is the unique regular measure extension of μ. Then, $\nu \in M_R^\sigma(\mathscr{L}_2)$.

Proof:

$\mathscr{L}_2$ c.p. $\Rightarrow \forall\ B_n \in \mathscr{L}_2,\ B_n \downarrow \emptyset,\ \exists\ \tilde{B}_n \in \mathscr{L}_2,\ B_n \subset \tilde{B}_n' \downarrow \emptyset$

Theorem 3.5 $\Rightarrow \nu \in M_\sigma(\mathscr{L}_2'),\ \nu(\tilde{B}_n') \to 0 \Rightarrow \nu(B_n) \to 0 \quad \forall\ B_n \in \mathscr{L}_2$

$\therefore$ ν is σ-smooth on $\mathscr{L}_2$, and since ν is regular on $\mathscr{L}_2$, $\nu \in M_R^\sigma(\mathscr{L}_2)$.

∎

We now give two applications of the results on coallocation lattices to topological spaces.

1) Measures on a locally compact Hausdroff space

Let X be a *locally compact Hausdroff space* and $\mathscr{L}_1 = K_0$ is the collection of all *compact* G_δ-sets, while $\mathscr{L}_2 = K$ is the collection of all compact sets. Note that in this case, X does not belong to either K_0 or K, unless X is compact. Then $K_0 \subset K$, and it can be shown that K coallocates K_0. For any $\mu \in M_R(K_0)$, μ is σ-smooth, because K_0 is compact. Thus, $\mu \in M_R^\sigma(K_0)$. By the coallocation theorem, we can extend μ uniquely to a regular measure ν which is also σ-smooth, because K is compact. Hence, $\nu \in M_R^\sigma(K)$.

2) Measure enlargement from zero sets to closed sets

Suppose X is a *countably paracompact* and *normal* topological space. Let $\mathscr{L}_1 = \mathfrak{Z}$ (*zero* sets) and $\mathscr{L}_2 = \mathfrak{F}$ (*closed* sets). That is, $\mathfrak{F}$ is c.p. normal. $\mathfrak{Z} \subset \mathfrak{F}$ because all zero sets are closed G_δ-sets, and disjoint closed sets can be separated by disjoint zero sets. Therefore, $\mathfrak{Z}$ is c.p. and normal. Thus, $\mathfrak{Z}$ coseparates $\mathfrak{F}$. Hence $\mathfrak{F}$ coallocates $\mathfrak{Z}$ [Theorem 2.3].

Let $\mu \in M_R(\mathfrak{Z})$. Then by Theorem 3.2, there exists a unique regular measure extension $\nu \in M_R(\mathfrak{F})$. Theorem 3.1(7)[a] implies ν on all open sets is $\mathfrak{Z}$-regular.

Suppose $\mu \in M_R^\sigma(\mathfrak{Z})$. By Theorem 3.5, the unique regular measure extension is $\nu \in M_R(\mathfrak{F}) \cap M_\sigma(\mathfrak{F}')$. Now $\mathfrak{F}$ is c.p., hence $\nu \in M_\sigma(\mathfrak{F})$ [Theorem 2.4]. Then, $\nu \in M_R^\sigma(\mathfrak{F})$. This is the result of Marik [5].

4. NORMAL LATTICES

In this section, we give further characterization of normal lattices and further consequences of a lattice being normal in terms of associated measures on the generalized algebra.

Definition 4.1

Let $\mathscr{L}$ be a lattice of subsets of X, and $\mu \in M(\mathscr{L})$. $\forall E \subset X$, define

$$\mu'(E) \equiv \inf \{ \mu(\tilde{L}') : E \subset \tilde{L}', \tilde{L} \in \mathscr{L} \}$$

$$\mu''(E) \equiv \inf \{ \textstyle\sum_{n=1}^{\infty} \mu(\tilde{L}_n') : E \subset \bigcup_{n=1}^{\infty} \tilde{L}_n', \tilde{L}_n \in \mathscr{L} \}$$

$$\mu_*(E) \equiv \sup \{ \mu(\tilde{L}) : E \supset \tilde{L} \in \mathscr{L} \}$$

$$\mu\hat{}(E) \equiv \inf \{ \mu_*(\tilde{L}') : E \subset \tilde{L}', \tilde{L} \in \mathscr{L} \}$$

It is clear that if $\mu \in M_R(\mathscr{L})$, then $\mu = \mu'$ on $A(\mathscr{L})$.

THEOREM 4.1

Let $\mu,\nu \in M(\mathscr{L})$, such that $\mu(X) = \nu(X)$. Then

$$\mu \le \nu \text{ on } \mathscr{L} \iff \mu \le \nu \le \nu' \le \mu' \text{ on } \mathscr{L}$$

Proof:

It is obvious that $\mu \le \nu$ on $\mathscr{L} \iff \nu \le \mu$ on $\mathscr{L}'$. Let $E \subset X$ s.t. $E \subset \tilde{L}'$, $\tilde{L} \in \mathscr{L}$, $\nu(\tilde{L}') \le \mu(\tilde{L}')$. Taking inf,

$$\inf\{\ \nu(\tilde{L}') : \tilde{L}' \in \mathscr{L}'\ \} \le \inf\{\ \mu(\tilde{L}') : \tilde{L}' \in \mathscr{L}'\ \},$$

$\therefore\ \nu'(E) \le \mu'(E)$. In particular, $E \in \mathscr{L} \Rightarrow \nu' \le \mu'$ on $\mathscr{L}$. Hence, $\mu \le \nu \le \nu' \le \mu'$ on $\mathscr{L}$.

∎

THEOREM 4.2

Suppose $\forall \mu \in I(\mathscr{L})$, $L_1, L_2 \in \mathscr{L}$,

$$\mu'(L_1) = 1 \text{ and } \mu'(L_2) = 1 \quad \Rightarrow \quad \mu'(L_1 \cap L_2) = 1$$

Then, $\mathscr{L}$ is *normal*.

Proof:

Suppose $\mathscr{L}$ is not normal. Then

$\exists\, \mu \in I(\mathscr{L})$, $\nu_1, \nu_2 \in I_R(\mathscr{L})$, s.t. $\mu \le \nu_1$ on $\mathscr{L}$, $\mu \le \nu_2$ on $\mathscr{L}$, but $\nu_1 \ne \nu_2$.

$\therefore\ \exists\, L_1, L_2 \in \mathscr{L}$, $L_1 \cap L_2 = \emptyset$,

$$\nu_1(L_1) = 1,\ \nu_2(L_1) = 0 \quad \text{and} \quad \nu_1(L_2) = 0,\ \nu_2(L_2) = 1$$

Now if $L_1 \subset \tilde{L}_1'$, $\tilde{L}_1 \in \mathscr{L}$, then $\nu_1(\tilde{L}_1') = 1$. Since $\mu \le \nu_1$ on $\mathscr{L} \iff \nu_1 \le \mu$ on $\mathscr{L}'$, we have $\mu(\tilde{L}_1') = 1 \Rightarrow \mu'(L_1) = 1$.

Similarly, if $L_2 \subset \tilde{L}_2'$, $\tilde{L}_2 \in \mathscr{L}$, then $\mu'(L_2) = 1$.

Then, by assumption, $\mu'(L_1 \cap L_2) = 1$. But $L_1 \cap L_2 = \emptyset$, $\therefore\ \mu'(L_1 \cap L_2) = 0$ gives a contradiction. Consequently, $\mathscr{L}$ is normal.

∎

THEOREM 4.3

Let $\nu \in M_R(\mathscr{L})$, $\rho \in M(\mathscr{L})$, s.t. $\nu(X) = \rho(X)$ and on $\mathscr{L}'$, $\nu \le \rho \in M_R(\mathscr{L}')$. Then

(1) $\rho \le \nu = \nu' \le \rho'$ on $\mathscr{L}$

(2) $\mathscr{L}$ is *normal* $\Rightarrow$ $\nu = \nu' = \rho'$ on $\mathscr{L}$.

Proof:

(1) $\nu \le \rho$ on $\mathscr{L}' \iff \rho \le \nu$ on $\mathscr{L}$, and $\nu \in M_R(\mathscr{L}) \Rightarrow \nu = \nu'$. Hence by Theorem 4.1, $\rho \le \nu = \nu' \le \rho'$ on $\mathscr{L}$.

(2) Suppose $\mathscr{L}$ is normal and $\exists\, L \in \mathscr{L}$ s.t. $\nu(L) < \rho'(L)$.

$\nu \in M_R(\mathscr{L}) \Rightarrow \forall \varepsilon > 0,\ \exists\, \tilde{L} \in \mathscr{L},\ \tilde{L} \subset L',\ \nu(\tilde{L}) + \varepsilon > \nu(L')$

$$\therefore \quad \nu(\tilde{L}') < \nu(L) + \varepsilon \quad \text{and} \quad L \cap \tilde{L} = \emptyset$$

By normality, $\exists\, L_a, L_b \in \mathscr{L}$, s.t. $L \subset L_a'$, $\tilde{L} \subset L_b'$, $L_a' \cap L_b' = \emptyset$

$\therefore \quad L \subset L'_a \subset L_b \subset \tilde{L}'$

$\nu(L) < \rho'(L) \le \rho'(L'_a) \underset{\mathscr{L}'}{=} \rho(L'_a) \le \rho(L_b) \underset{\mathscr{L}}{\le} \nu(L_b) \le \nu(\tilde{L}') < \nu(L) + \varepsilon$

$\Rightarrow \rho'(L) \le \nu(L)$ gives a contradiction. $\therefore \nu = \nu' = \rho'$ on $\mathscr{L}$. ∎

THEOREM 4.4

Let $\mathscr{L}$ be a lattice of subsets of X, and $\mu \in M_\sigma(\mathscr{L})$. Then,

(1) $\mu'' \le \mu'$ *everywhere*

(2) $\mu' = \mu$ on $\mathscr{L}'$

(3) $\mu \le \mu'' \le \mu'$ on $\mathscr{L}$

(4) $\mu(X) = \mu''(X) = \mu'(X)$

(5) $\mu_*(L') + \mu'(L) = \mu(X), \quad \forall L \in \mathscr{L}$

(6) If $\mathscr{L}$ is *normal*, then $\mu \le \mu'' \le \mu' = \hat{\mu}$ *on* $\mathscr{L}$

(7) If $\mathscr{L}$ is δ-*normal*, then $\mu'' = \mu'$ *on* $\mathscr{L}$.

NOTE : The condition $\mu \in M_\sigma(\mathscr{L})$ is imposed, because when μ is a 0-1 measure and if μ is not σ-smooth, then $\mu'' \equiv 0$.

Proof:

(1) By definition of μ'', the inf encompasses more sets than that of μ', hence $\mu'' \le \mu'$ everywhere.

(2) Take $E = \tilde{L}' \in \mathscr{L}'$, $\therefore \mu' = \mu$ on $\mathscr{L}'$. In particular,

$$\mu'(X) = \mu(X) \qquad \text{.................... [i]}$$

(3) From (1) and [i], we have $\mu''(X) \le \mu(X)$. We now show that $\mu''(X) = \mu(X)$. For suppose $X = \bigcup_{i=1}^\infty L_i'$, pairwise disjoint $L_i' \in \mathscr{L}'$, and $\sum_{i=1}^\infty \mu(L_i') < \mu(X)$, but $\sum_{i=1}^\infty \mu(L_i') = \lim_{n\to\infty} \sum_{i=1}^n \mu(L_i')$ $\ge \lim_{n\to\infty} \mu(\bigcup_{i=1}^n L_i') = \mu(X)$.

Since $\bigcup_{i=1}^n L_i' \in \mathscr{L}'$ and $\bigcup_{i=1}^n L_i' \uparrow X$, or $\bigcap_{i=1}^\infty L_i \downarrow \emptyset$, also $\mu \in M_\sigma(\mathscr{L})$.

Taking the inf of the above, we have $\mu''(X) \ge \mu(X)$.

Consequently, $\mu''(X) = \mu(X)$ .... [ii]

Now suppose $\exists\, L \in \mathscr{L}$, $\mu(L) > \mu''(L)$,

$$\begin{aligned} \mu''(X) = \mu''(L \cup L') &\le \mu''(L) + \mu''(L') \\ &\le \mu''(L) + \mu(L') && (\because \mu'' \le \mu \text{ on } \mathscr{L}') \\ &< \mu(L) + \mu(L') && (\text{ by assumption}) \\ &= \mu(X) && \text{contradicting [ii]} \end{aligned}$$

$\therefore \mu \le \mu''$ on $\mathscr{L}$. Together with (1), we have $\mu \le \mu'' \le \mu'$ on $\mathscr{L}$.

(4) [i] and [ii] $\Rightarrow \mu(X) = \mu''(X) = \mu'(X)$.

(5) $\forall \varepsilon > 0, \quad \mu_*(L') - \varepsilon < \mu(\hat{L}) \quad$ where $L' \supset \hat{L} \in \mathscr{L}$

$\therefore \quad \mu(X) - \mu_*(L') > \mu(\hat{L}') - \varepsilon$

Taking inf, we have

$$\mu(X) - \mu_*(L') \geq \mu'(L) \quad \text{..................................} [iii]$$

If $L' \supset \hat{L}, \quad \mu_*(L') \geq \mu(L') \geq \mu(\hat{L})$ since $\mu_* \leq \mu$ on $\mathscr{L}'$

$\therefore \quad \mu(X) - \mu_*(L') \leq \mu(X) - \mu(\hat{L}) = \mu(\hat{L}')$

Taking inf, we have

$$\mu(X) - \mu_*(L') \leq \inf\{\mu(\hat{L}') : L \subset \hat{L}', \hat{L} \in \mathscr{L}\} \equiv \mu'(L) \quad \text{.....} [iv]$$

[iii] and [iv] $\Rightarrow \mu(X) - \mu_*(L') = \mu'(L) \quad \forall L \in \mathscr{L}$.

(6) L is normal, then Corollary 3.1(2) and (5) imply $\mu\hat{}(L) = \mu'(L) \quad \forall L \in \mathscr{L}$.

Now $\mu' = \mu\hat{}$ on $\mathscr{L}$, and from (3), $\mu \leq \mu'' \leq \mu' = \mu\hat{}$ on $\mathscr{L}$.

(7) From (1), $\mu'' \leq \mu'$ everywhere. Suppose $\exists L \in \mathscr{L}$, s.t. $\mu''(L) < \mu'(L)$

Then $L \subset \bigcup_{n=1}^{\infty} L_n'$, $L_n \in \mathscr{L}$. $\mathscr{L}$ is $\delta \Rightarrow \bigcap_{n=1}^{\infty} L_n \in \mathscr{L}$.

Let $D \equiv \bigcap_{n=1}^{\infty} L_n$, $L \cap D = \emptyset$. $\mathscr{L}$ is normal $\Rightarrow \exists A, B \in \mathscr{L}$, s.t.

$$L \subset A', \ D \subset B', \ A' \cap B' = \emptyset \quad \therefore \quad L \subset A' \subset B \subset D' \equiv \bigcup_{n=1}^{\infty} L_n'$$

$$\mu'(L) \leq \mu'(A') \underset{\text{from (2)}}{=} \mu(A') \leq \mu(B) \underset{\text{from (3)}}{\leq} \mu''(B) \leq \textstyle\sum_{n=1}^{\infty} \mu(L_n')$$

Taking inf, and since $L \subset \bigcup_{n=1}^{\infty} L_n'$,

$$\mu'(L) \leq \inf\{\textstyle\sum_{n=1}^{\infty} \mu(L_n') : L \subset \sum_{n=1}^{\infty} L_n', \ L_n \in \mathscr{L}\} \equiv \mu''(L)$$

gives a contradiction. $\therefore \mu''(L) = \mu'(L)$, or $\mu'' = \mu'$ on $\mathscr{L}$. ∎

THEOREM 4.5

Let $\mathscr{L}$ be a lattice of subsets of X, and let $\mu \in M_\sigma(\mathscr{L})$, $\rho \in M(\mathscr{L})$, s.t. $\mu \leq \rho$ on $\mathscr{L}$, $\mu(X) = \rho(X)$.

If $\mathscr{L}$ is *countably paracompact* and *normal*, then $\rho \in M_\sigma(\mathscr{L})$.

Proof:

Let $L_n \downarrow \emptyset$, $L_n \in \mathscr{L}$, $\forall n$

$\mathscr{L}$ c.p. $\Rightarrow \exists \tilde{L}_n \in \mathscr{L}$, $L_n \subset \tilde{L}_n' \downarrow \emptyset \quad \therefore L_n \cap \tilde{L}_n = \emptyset$

$\mathscr{L}$ normal $\Rightarrow \exists A_n, B_n \in \mathscr{L}$, $L_n \subset A_n'$, $\tilde{L}_n \subset B_n'$, $A_n' \cap B_n' = \emptyset$.

Or, $L_n \subset A_n' \subset B_n \subset \tilde{L}_n' \downarrow \emptyset$, $\therefore \rho(L_n) \leq \rho(A_n') \leq \mu(A_n') \leq \mu(B_n) \rightarrow 0$

(one may assume, with the loss of generality, $B_n \downarrow$).

($\because \rho \leq \mu$ on $\mathscr{L}'$; $B_n \downarrow \emptyset$ and $\mu \in M_\sigma(\mathscr{L})$) $\therefore \rho(L_n) \rightarrow 0$, or $\rho \in M_\sigma(\mathscr{L})$. ∎

THEOREM 4.6

Suppose $\mathscr{L}_1 \subset \mathscr{L}_2$, and $\mathscr{L}_1$ *separates* $\mathscr{L}_2$. Then,

$$\mathscr{L}_1 \ \textit{normal} \Leftrightarrow \mathscr{L}_2 \ \textit{normal}.$$

Proof:

"$\Rightarrow$" Suppose $\mathscr{L}_1$ is normal.

Let $\mu \in I(\mathscr{L}_2)$, $\nu_a, \nu_b \in I_R(\mathscr{L}_2)$, s.t. $\mu \le \nu_a$ on $\mathscr{L}_2$, $\mu \le \nu_b$ on $\mathscr{L}_2$.

Then $\mu| \in I(\mathscr{L}_1)$, $\nu_a|, \nu_b| \in I_R(\mathscr{L}_1)$, and $\mu| \le \nu_a|$ on $\mathscr{L}_1$, $\mu| \le \nu_b|$ on $\mathscr{L}_1$. $\mathscr{L}_1$ normal $\Leftrightarrow$ $\nu_a| = \nu_b|$ [Theorem 2.2].

Extend $\nu_a|$ and $\nu_b|$ to $\mathscr{L}_2 \Rightarrow \nu_a = \nu_b \Leftrightarrow \mathscr{L}_2$ normal,

$\because$ $\mathscr{L}_1$ separates $\mathscr{L}_2$, the extension is unique.

"$\Leftarrow$" Suppose $\mathscr{L}_2$ is normal.

Let $\mu \in I(\mathscr{L}_1)$, $\nu_a, \nu_b \in I_R(\mathscr{L}_1)$, s.t. $\mu \le \nu_a$ on $\mathscr{L}_1$, $\mu \le \nu_b$ on $\mathscr{L}_1$.
Extend μ to $\lambda \in I(\mathscr{L}_2)$, and ν_a, ν_b to $\tau_a, \tau_b \in I_R(\mathscr{L}_2)$, respectively. We now show that $\lambda \le \tau_a$ on $\mathscr{L}_2$, and $\lambda \le \tau_b$ on $\mathscr{L}_2$.
For suppose $\exists\ L_2 \in \mathscr{L}_2$ s.t. $\lambda(L_2) = 1$ but $\tau_a(L_2) = 0$. Then $\tau_a(L_2') = 1$. But $\tau_a \in I_R(\mathscr{L}_2)$, $\exists\ \tilde{L}_2 \in \mathscr{L}_2$, s.t. $\tilde{L}_2 \subset L_2'$, $\tau_a(\tilde{L}_2) = 1$
Since $\mathscr{L}_1$ separates $\mathscr{L}_2 \Rightarrow \exists\ L_1 \in \mathscr{L}_1$, s.t. $L_2 \subset L_1 \subset \tilde{L}_2'$

$$\therefore 1 = \lambda(L_2) \le \lambda(L_1) \underset{\mathscr{L}_1}{=} \mu(L_1) \underset{\mathscr{L}_1}{\le} \nu_a(L_1) \underset{\mathscr{L}_1}{=} \tau_a(L_1) \le \tau_a(\tilde{L}_2')$$

Thus, $\tau_a(\tilde{L}_2') = 1$ or $\tau_a(\tilde{L}_2) = 0$ contradicting $\tau_a(\tilde{L}_2) = 1$
$\therefore$ $\lambda \le \tau_a$ on $\mathscr{L}_2$. Similarly, $\lambda \le \tau_b$ on $\mathscr{L}_2$. Since $\mathscr{L}_2$ is normal, $\tau_a = \tau_b$ $\therefore \tau_a| = \tau_b|$, i.e. $\nu_a = \nu_b$ $\Leftrightarrow \mathscr{L}_1$ is normal.

∎

THEOREM 4.7

Suppose $\mathscr{L}_1 \subset \mathscr{L}_2$, and $\mu \in M_R(\mathscr{L}_1)$, $\nu \in M_R(\mathscr{L}_2)$, s.t. $\mu(X) = \nu(X)$, $\nu|_{\mathscr{L}_1} = \mu$. Then

$\mathscr{L}_1$ separates $\mathscr{L}_2$ $\Rightarrow$ ν is $\mathscr{L}_1$-regular on $\mathscr{L}_2'$.

Proof:

$\nu \in M_R(\mathscr{L}_2)$, $\therefore\ \forall\ L_2' \in \mathscr{L}_2'$, $\nu(L_2') = \sup\{\ \nu(\tilde{L}_2) : L_2' \supset \tilde{L}_2 \in \mathscr{L}_2\ \}$

$\forall\ \varepsilon > 0$, $L_2' \supset \tilde{L}_2 \in \mathscr{L}_2$, $\nu(L_2') < \nu(\tilde{L}_2) + \varepsilon$

$L_2 \cap \tilde{L}_2 = \emptyset$, and $\mathscr{L}_1$ separates $\mathscr{L}_2 \Rightarrow$

$\exists\ L_1, \tilde{L}_1 \in \mathscr{L}_1$, s.t. $L_2 \subset L_1$, $\tilde{L}_2 \subset \tilde{L}_1$, $L_1 \cap \tilde{L}_1 = \emptyset$

$$\begin{aligned} \nu(L_2') &< \nu(\tilde{L}_2) + \varepsilon \\ &\le \nu(\tilde{L}_1) + \varepsilon \quad (\ \tilde{L}_2 \subset \tilde{L}_1\ ) \\ &= \mu(\tilde{L}_1) + \varepsilon \quad (\ \nu|_{\mathscr{L}_1} = \mu\ ) \end{aligned}$$

Taking sup, $\nu(L_2') = \sup\{\ \mu(\tilde{L}_1) : L_2' \supset \tilde{L}_1 \in \mathscr{L}_1\ \}$, $\forall\ L_2' \in \mathscr{L}_2'$

i.e. ν is *$\mathscr{L}_1$-regular on $\mathscr{L}_2'$*.

∎

ACKNOWLEDGEMENT

I express my gratitude to my teacher and dissertation advisor, Professor George Bachman of Polytechnic University (Brooklyn, NY), for his guidance, encouragement, and proof reading of this paper. I would also like to thank the two Referees for their valuable comments.

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