Internat. J. Math. & Math. Sci.
VOL. 16 NO. 1 (1993) 49-60

ON ALEXANDROV LATTICES

ALBERT GORELISHVILI

Department of Mathematics
New York Institute of Technology
Old Westbury, New York 11568

(Received October 10, 1991 and in revised form January 20, 1992)

ABSTRACT. By an Alexandrov lattice we mean a δ normal lattice of subsets of an abstract set X, such that the set of $\mathcal{L}$-regular countably additive bounded measures is sequentially closed in the set of $\mathcal{L}$-regular finitely additive bounded measures on the algebra generated by $\mathcal{L}$ with the weak topology.

For a pair of lattices $\mathcal{L}_1 \subset \mathcal{L}_2$ in X sufficient conditions are indicated to determine when $\mathcal{L}_1$ Alexandrov implies that $\mathcal{L}_2$ is also Alexandrov and vice versa. The extension of this situation is given where $T: X \to Y$ and $\mathcal{L}_1$ and $\mathcal{L}_2$ are lattices of subsets of X and Y respectively and T is $\mathcal{L}_1 - \mathcal{L}_2$ continuous.

KEY WORDS AND PHRASES. Lattices, topological measures, Wallman topology.
1991 AMS SUBJECT CLASSIFICATION CODES. Primary 28A32, 46E27, 28C15, 46G12.

1. INTRODUCTION.

We adhere for the most part to the basic terminology of A. Alexandrov [1] (see also H. Bergstrom [6]). Let X be an abstract set, and $\mathcal{L}$ a lattice of subsets of X. $MR(L)$ denotes the $\mathcal{L}$-regular finitely additive bounded measures on $\mathfrak{A}(\mathcal{L})$, the algebra generated by $\mathcal{L}$, and $MR(\sigma, \mathcal{L})$ those elements of $MR(\mathcal{L})$ that are countably additive. We assume without loss of generality that all measures are non-negative.

A fundamental theorem of A. Alexandrov states that if $\mathcal{L}$ is δ normal and complement generated (i.e., completely normal), then $\mu_n \in MR(\sigma, \mathcal{L})$ and $\mu_n \xrightarrow{w} \mu$ (i.e., converges weakly) implies that $\mu \in MR(\sigma, \mathcal{L})$.

In general we will call lattices for which this is true Alexandrov lattices, and our major concern in this paper is in determining further type lattices which are Alexandrov. In particular, we investigate the interrelationships between a pair of lattices $\mathcal{L}_1 \subset \mathcal{L}_2$ in X and determine conditions when $\mathcal{L}_1$ Alexandrov implies $\mathcal{L}_2$ Alexandrov and conversely, and then extend this to the situation where $T: X \to Y$ and $\mathcal{L}_1, \mathcal{L}_2$ are lattices of subsets of X and Y respectively and T is $\mathcal{L}_1 - \mathcal{L}_2$ continuous. It is well known (see [5] that if $\mu \in MR(\mathcal{L})$, then μ induces measures $\hat{\mu}$ and $\tilde{\mu}$ on the associated Wallman space $IR(\mathcal{L})$ and also a measure μ' on the space $IR(\sigma, \mathcal{L})$ (see below for definitions), and we investigate how weak convergence: $\mu_n \xrightarrow{w} \mu$ in general is reflected over to these induced measures. This enables us to give alternative proofs of important results of Kirk and Crenshaw [8], who have also investigated certain aspects of topological measure theory in the Alexandrov framework.

We begin with certain notations and terminology which will be used throughout the paper, and then set up the general Alexandrov framework of reference. The associated Wallman space is then investigated, enabling us to readily generalize results of Varadarajan [9] and obtain in a different manner results of Kirk and Crenshaw. Finally, in the last section we investigate Alexandrov lattices and extend Alexandrov's fundamental theorem.

Our notation and terminology is standard for the most part (see [1], [6], [8], [5]), and we collect it in the next section for the reader's convenience.

2. TERMS AND NOTATION.

In this section we introduce some basic terms, facts, and notation of topological measure theory used throughout this paper.

Let X be a set and $\mathcal{L}$ be any lattice of subsets of X. We shall always assume that $\emptyset$, $X \in \mathcal{L}$. The following notation is used here: N for the natural numbers, R for the real numbers, x for the general element of X, $\mathfrak{A}(\mathcal{L})$ for the smallest algebra containing $\mathcal{L}$, $\sigma(\mathcal{L})$ for the smallest σ-algebra containing $\mathcal{L}$. $\delta(\mathcal{L})$ is the set of all arbitrary intersections $\cap \mathcal{L}_i$ with $L_i \in \mathcal{L}$ and $\tau(\mathcal{L})$ is the set of all arbitrary intersections A_α with $A_\alpha \in \mathcal{L}$. $\mathcal{L}$ is complemented if $A \in \mathcal{L}$ implies $A' \in \mathcal{L}$ where $A' = X - A$. $\mathcal{L}'$ is the class of all complements of $\mathcal{L}$-sets, i.e., $\mathcal{L}' = \{L' : L \in \mathcal{L}\}$. $\mathcal{L}$ is complement generated if $A \in \mathcal{L}$ implies $A = \cap A'_i$ where $A_i \in \mathcal{L}$; $s(\mathcal{L})$ are the Souslin sets determined by $\mathcal{L}$.

$\mathcal{L}$ is separating or T_1 if for all $x, y \in X$, $x \neq y$ implies there exists an $A \in \mathcal{L}$ such that $x \in A$ and $y \notin A$. $\mathcal{L}$ is disjunctive if for any $A \in \mathcal{L}$ and $x \notin A$, there exists a $B \in \mathcal{L}$ such that $x \in B$ and $A \cap B = \emptyset$. $\mathcal{L}$ is Hausdorff or T_2 if for all $x, y \in X, x \neq y$ implies there exist $A, B \in \mathcal{L}$ such that $x \in A', y \in B'$ and $A' \cap B' = \emptyset$. $\mathcal{L}$ is regular if for every $x \in X$ and every $A \in \mathcal{L}$, $x \notin A$ implies there exist $B, C \in \mathcal{L}$ such that $x \in B', A \subset C'$ and $B' \cap C' = \emptyset$; $\mathcal{L}$ is normal if for all $A, B \in \mathcal{L}, A \cap B = \emptyset$ implies there exist $C, D \in \mathcal{L}$ such that $A \subset C', B \subset D'$, and $C' \cap D' = \emptyset$; $\mathcal{L}$ is strongly normal if it is δ, normal, disjunctive and separating. $\mathcal{L}$ is compact if any family of sets in $\mathcal{L}$ with the finite intersection property has a non-empty intersection. Similarly, we define $\mathcal{L}$ is countably compact (c.c.). $\mathcal{L}$ is countably paracompact (c.p.) if $A_n \in \mathcal{L}$ and $A_n \downarrow \emptyset$ imply there exist $B_n \in \mathcal{L}$ such that $A_n \subset B'_n$ and $B'_n \downarrow \emptyset$.

A function $f: X \to R \cup \{\pm\infty\}$ is $\mathcal{L}$-continuous if $f^{-1}(C) \in \mathcal{L}$ for every closed set $C \subset R \cup \{\pm\infty\}$. The set whose general element is a zero set of an $\mathcal{L}$-continuous function is denoted by $\mathfrak{Z}(\mathcal{L})$; $\mathfrak{Z} \in Z(\mathcal{L})$ iff $Z = f^{-1}(0)$ for some $\mathcal{L}$-continuous function f. A measure μ on $\mathfrak{A}(\mathcal{L})$ is a finitely additive bounded real-valued set function. $M(\mathcal{L})$ denotes the set of all measures on $\mathfrak{A}(\mathcal{L})$. A measure μ is said to be σ-smooth on $\mathcal{L}$ if $A_n \in \mathcal{L}$, $A_n \downarrow \emptyset$ implies $\mu(A_n) \to 0$. A measure $\mu \in M(\mathcal{L})$ is said to be $\mathcal{L}$-regular if for every $A \in \mathfrak{A}(\mathcal{L})$ and every $\varepsilon > 0$, there exists an $L \in \mathcal{L}$ such that $L \subset A$ and $|\mu(A) - \mu(L)| < \varepsilon$. The set whose general element is an $\mathcal{L}$-regular measure on $\mathfrak{A}(\mathcal{L})$ is denoted by $MR(\mathcal{L})$, and the set whose general element is an element of $M(\mathcal{L})$, which is σ-smooth on $\mathcal{L}$, is denoted by $M(\sigma, \mathcal{L})$. Moreover, we use the notation $MR(\sigma, \mathcal{L}) = MR(\mathcal{L}) \cap M(\sigma, \mathcal{L})$. The set of all measures μ such that $\mu(A) = \{0, 1\}$ for every $A \in \mathfrak{A}(\mathcal{L})$ and $\mu(X) = 1$ is denoted by $I(\mathcal{L})$. The set of all $\{0, 1\}$-valued $\mathcal{L}$-regular measures is denoted by $IR(\mathcal{L})$, i.e., $IR(\mathcal{L}) = I(\mathcal{L}) \cap MR(\mathcal{L})$. The Dirac measure (concentrated) at x is denoted by μ_x. For $\mu \in M(\mathcal{L})$ the support of μ is defined and denoted by $S(\mu) = \cap \{L \in \mathcal{L} : |\mu|(L) = |\mu|(X)\}$, where $|\mu|$ denoted the variation of measure μ. $\mathcal{L}$ is said to be replete if for every $\mu \in IR(\sigma, \mathcal{L})$ we have $S(\mu) \neq \emptyset$, where $IR(\sigma, \mathcal{L}) = IR(\mathcal{L}) \cap I(\sigma, \mathcal{L})$. A measure $\mu \in MR(\mathcal{L})$ is τ-smooth if $L_\alpha \downarrow \emptyset$ implies $\mu(L_\alpha) \to 0$ for any net $\{L_\alpha\}$ in $\mathcal{L}$. The set of τ-smooth regular measures is denoted by $MR(\tau, \mathcal{L})$.

Since any measure $\mu \in M(\mathcal{L})$ splits into its non-negative and non-positive parts μ^+ and μ^- respectively , w.l.o.g. we shall work with non-negative measures.

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two lattices of subsets of X. Throughout this paper we shall assume that $\mathcal{L}_1 \subset \mathcal{L}_2$. The following describe relationships between $\mathcal{L}_1$ and $\mathcal{L}_2$.

$\mathcal{L}_1$ semiseparates (s.s) $\mathcal{L}_2$ if for every $A \in \mathcal{L}_1$ and $B \in \mathcal{L}_2$, $A \cap B = \emptyset$ implies there exists a $C \in \mathcal{L}_1$ such that $B \subset C$ and $A \cap C = \emptyset$; $\mathcal{L}_1$ separates $\mathcal{L}_2$ if for all A, $B \in \mathcal{L}_2$, $A \cap B = \emptyset$ implies there exist $C, D \in \mathcal{L}_1$ such that $A \subset C, B \subset D$ and $C \cap D = \emptyset$. $\mathcal{L}_1$ coseparates $\mathcal{L}_2$ if for all A, $B \in \mathcal{L}_2$, $A \cap B = \emptyset$ implies there exist C, $D \in \mathcal{L}_1$ such that $A \subset C'$, $B \subset D'$ and $C' \cap D' = \emptyset$. Clearly, $\mathcal{L}_1$ coseparates $\mathcal{L}_2$ implies $\mathcal{L}_1$ separates $\mathcal{L}_2$, and $\mathcal{L}_1$ separates $\mathcal{L}_2$ implies $\mathcal{L}_1$ semiseparates $\mathcal{L}_2$. $\mathcal{L}_2$ is $\mathcal{L}_1$ countably paracompact (c.p.) if for every $A_n \in \mathcal{L}_2$, $A_n \downarrow \emptyset$ imply there exist $B_n \in \mathcal{L}_1$ such that $A_n \subset B'_n$ and $B'_n \downarrow \emptyset$. $\mathcal{L}_2$ is $\mathcal{L}_1$ countably bounded (c.b.) if for all $A_n \in \mathcal{L}_2$, $A_n \downarrow \emptyset$ imply there exist $B_n \in \mathcal{L}_1$ such that $A_n \subset B_n$ and $B_n \downarrow \emptyset$.

For the restriction of $\nu \in MR(\mathcal{L}_2)$ to $\mathfrak{A}(\mathcal{L}_1)$ we adopt the notation $\nu|_{\mathcal{L}_1}$ or, simply, $\nu|$. Note that if $\nu \in MR(\mathcal{L}_2)$ and if $\mathcal{L}_1$ s.s. $\mathcal{L}_2$, then $\nu| \in MR(\mathcal{L}_1)$.

We conclude this section with the following general extension theorem.

THEOREM 2.1 [4]. Let $\mathcal{L}_1 \subset \mathcal{L}_2$ be two lattices of subsets of X. Then any measure $\mu \in MR(\mathcal{L}_1)$ can be extended to a ν, $\nu \in MR(\mathcal{L}_2)$, and if $\mathcal{L}_1$ separates $\mathcal{L}_2$ then ν is unique. If $\mu \in MR(\sigma_1, \mathcal{L}_1)$ and $\mathcal{L}_2$ is $\mathcal{L}_1$ countably paracompact or countably bounded, then $\nu \in MR(\sigma, \mathcal{L}_2)$.

3. WALLMAN SPACES.

Let X be an abstract set and $\mathcal{L}$ be a lattice of subsets of X with $\emptyset$, $X \in \mathcal{L}$. In this section we review some facts pertaining to the Wallman spaces $IR(\mathcal{L})$ and $IR(\sigma, \mathcal{L})$, and we introduce measures induced by $\mu \in M(\mathcal{L})$ on various algebras generated by lattices in these spaces.

We assume for convenience throughout that $\mathcal{L}$ is a disjunctive lattice, although this is not necessary in all statements that follow.

Define $W(A) = \{\mu \in IR(\mathcal{L}): \mu(A) = 1\}$, for $A \in \mathfrak{A}(\mathcal{L})$.

PROPOSITION 3.1. If $\mathcal{L}$ is a disjunctive lattice, then $\forall A, B \in \mathfrak{A}(\mathcal{L})$ we have

i) $W(A \cap B) = W(A) \cap W(B)$

ii) $W(A \cup B) = W(A) \cup W(B)$

iii) $A \subset B \Leftrightarrow W(A) \subset W(B)$

iv) $W(A)' = W(A')$

v) $W(\mathfrak{A}(\mathcal{L})) = \mathfrak{A}(W(\mathcal{L}))$

Consequently,

$W(\mathcal{L}) = \{W(A): A \in \mathcal{L}\}$ is a disjunctive lattice.

Note that if $\mathcal{L}$ is separating and disjunctive, then the closure in $IR(\mathcal{L})$ of $L \in \mathcal{L}$ is given by $\overline{L} = \cap\{W(A): L \subset W(A), A \in \mathcal{L}\} = W(L)$.

PROPOSITION 3.2. To each $\mu \in M(\mathcal{L})$, there corresponds a $\widehat{\mu} \in M(W(\mathcal{L}))$ defined by $\widehat{\mu}(W(A)) = \mu(A)$, $A \in \mathfrak{A}(\mathcal{L})$ such that

a) $\widehat{\mu}$ is well-defined.

b) $\widehat{\mu} \in M(W(\mathcal{L}))$,

c) if $\nu \in M(W(\mathcal{L}))$, then $\nu = \widehat{\mu}$, for some $\mu \in M(\mathcal{L})$,

d) $\mu \in MR(\mathcal{L})$ if and only if $\widehat{\mu} \in MR(W(\mathcal{L}))$.

PROOF.

a) Since $\mathcal{L}$ is disjunctive, we have $W(A) = W(B) \Rightarrow A = B \Rightarrow \mu(A) = \mu(B) \Rightarrow \widehat{\mu}(W(A)) = \widehat{\mu}(W(B))$

b) If $W(A) \cap W(B) = W(A \cap B) = \emptyset$, then $A \cap B = \emptyset$ (because $\mathcal{L}$ is disjunctive) and $\widehat{\mu}(\emptyset) = \mu(\emptyset) = 0$. $\widehat{\mu}(W(A)) \cup W(B)) = \mu(A \cup B) = \mu(A) + \mu(B) = \widehat{\mu}(W(A)) + \widehat{\mu}(W(B)), \widehat{\mu} \in M(W(\mathcal{L}))$

c) We have $\mu_1 \neq \mu_2 \Rightarrow \mu_1(A) \neq \mu_2(A)$ for some $A \in \mathcal{L}$. Therefore, $\widehat{\mu}_1(W(A)) \neq \widehat{\mu}_1(W(A))$; hence, $\mu_1 \neq \mu_2$. Suppose $\nu \in M(W(\mathcal{L}))$. Define μ on $\mathfrak{A}(\mathcal{L})$ by $\mu(A) = \nu(W(A))$ for all $A \in \mathfrak{A}(\mathcal{L})$. Then, μ is well-defined and $\nu = \widehat{\mu}$.

d) It suffices to show that $\hat{\mu} \in MR(W(\mathcal{L}))$ implies $\mu \in MR(\mathcal{L})$. Since μ is $\mathcal{L}$-regular, $\mu(A) \sim \mu(L)$ and $A \in \mathfrak{A}(\mathcal{L}), A \supset L$. However, $\mu(L) = \hat{\mu}(W(L))$. Therefore, $W(L) \subset W(A)$. Hence, $\mu \in MR(\mathcal{L})$.

We can define a closed set topology on $IR(\mathcal{L})$ by taking the closed sets $\mathcal{F} = \tau W(\mathcal{L})$. This generates the Wallman topology (W-top.). The topological space $\{IR(\mathcal{L}), W\}$ is compact and T_1. It is T_2 if and only if $\mathcal{L}$ is normal.

Since $W(\mathcal{L})$ and $\tau W(\mathcal{L})$ are compact lattices, $W(\mathcal{L})$ separates $\tau W(\mathcal{L})$. Therefore, if $\mu \in MR(\mathcal{L})$, then $\hat{\mu} \in MR(W(\mathcal{L}))$ and by Theorem 2.1 has a unique extension to $\tilde{\mu} \in MR(\tau W(\mathcal{L}))$. We note that since $W(\mathcal{L})$ and $\tau W(\mathcal{L})$ are compact lattices, $\hat{\mu}$ and $\tilde{\mu}$ are not only σ-smooth on their respective lattices, but also τ-smooth. Since both $\hat{\mu}$ and $\tilde{\mu}$ are countably additive, we can extend them uniquely to $\sigma(W(\mathcal{L}))$ and $\sigma(\tau W(\mathcal{L}))$ respectively and continue to denote the extensions by $\hat{\mu}$ and $\tilde{\mu}$. Note that $\hat{\mu}$ is $\delta W(\mathcal{L})$ regular on $\sigma(W(\mathcal{L}))$ while μ is still $\tau W(\mathcal{L})$ regular on $\sigma(\tau W(\mathcal{L}))$.

We now summarize some smoothness properties of μ in terms of $\hat{\mu}$ and $\tilde{\mu}$ (further details can be found in [5]).

PROPOSITION 3.3. Let X be a set and let $\mathcal{L}$ be separating and disjunctive. If $\mu \in MR(\mathcal{L})$, then the following statements are equivalent:

1) $\mu \in MR(\sigma, \mathcal{L})$
2) $\hat{\mu}(\cap W(L_i)) = 0, \quad \cap W(L_i) \subset IR(\mathcal{L}) - X, \ L_i \downarrow, \ L_i \in \mathcal{L}$
3) $\hat{\mu}(\cap W(L_i)) = 0, \quad \cap W(L_i) \subset IR(\mathcal{L}) - IR(\sigma, \mathcal{L}), \ L_i \downarrow, \ L_i \in \mathcal{L}$
4) $\hat{\mu}^*(X) = \hat{\mu}(IR(\mathcal{L}))$, where $\hat{\mu}^*$ is the induced "outer" measure.

PROPOSITION 3.4. Let $\mathcal{L}$ be a separating and disjunctive lattice of subsets of X and let $\mu \in MR(\mathcal{L})$. The following statements are equivalent:

1) $\mu \in MR(\tau, \mathcal{L})$
2) $\tilde{\mu}$ vanishes on every W-closed set of $IR(\mathcal{L}) - X$
3) $\tilde{\mu}^*(X) = \tilde{\mu}(IR(\mathcal{L}))$

Under the same conditions on $\mathcal{L}$, when $\mu \in MR(\mathcal{L})$, we also have

PROPOSITION 3.5. The following statements are true:

1) $\tilde{\mu}$ on $\mathcal{O} = (\tau W(\mathcal{L}))'$ is $W(\mathcal{L})$-regular
2) $\hat{\mu}^* = \tilde{\mu}$ on $\tau W(\mathcal{L})$

PROOF. Define $\hat{\mu}(W(L)) = \mu(L)$, $L \in \mathfrak{A}(\mathcal{L})$; $\tilde{\mu}$ is $\hat{\mu}$ extended to $\mathfrak{A}(\tau W(\mathcal{L}))$. Let $O \in (\tau W(\mathcal{L}))'$, i.e., O is W open. Since $\tilde{\mu} \in MR(\tau W(\mathcal{L}))$, there exists $F \in \mathcal{F}$, $F \subset O$ such that $\mu(O - F) < \varepsilon$. Assume that $F = \cap W(L_\alpha)$. Then, $F \cap O' = \emptyset$. Thus, $\cap W(L\alpha_k) \cap O' = \emptyset$. Hence, $W(\mathcal{L}) = W(\cap L_{\alpha_k}) \subset O$, $L \in \mathcal{L}$. Since $F = \cap W(L_\alpha)$, then $F \subset W(L)$ which implies that $\tilde{\mu}(O - W(L)) \leq \mu(O - F) < \varepsilon$, i.e., $\tilde{\mu}$ is $W(L)$-regular on $\mathcal{O} = (\tau W(\mathcal{L}))'$.

We now show that $\tilde{\mu}(F) = \hat{\mu}^*(F)$. Clearly, $\tau W(L)$ is a δ lattice. Also, $\tilde{\mu}(F) = \tilde{\mu}(\cap W(L_\alpha)) = inf\,\hat{\mu}(W(L_\alpha)) = inf\,\hat{\mu}(W(L_\alpha)) \geq \mu^*(F)$. Therefore, $\hat{\mu}^* \leq \tilde{\mu}$ on $\tau W(\mathcal{L})$. On the other hand, $\sigma(W(\mathcal{L})) \subset \sigma(\tau W(\mathcal{L}))$ and $\tilde{\mu}^* \leq \hat{\mu}^*$ everywhere. For $F \in \tau W(\mathcal{L}), \tilde{\mu}(F) = \tilde{\mu}^*(F) \leq \hat{\mu}^*(F)$. Hence, $\tilde{\mu} \leq \hat{\mu}^*$ on $\tau W(\mathcal{L})$. Finally, $\tilde{\mu} = \hat{\mu}^*$ on $\tau W(\mathcal{L})$ i.e., $\tilde{\mu} = \hat{\mu}_*$ on $(\tau W(\mathcal{L}))'$.

It is important to note that $\hat{\mu}$ is defined on zero sets of the W-topology. Namely, we have

PROPOSITION 3.6. Every zero set Z of a continuous function on $IR(\mathcal{L})$ is an element of $\sigma(W(\mathcal{L}))$.

PROOF. Z is compact, and also a G_δ set; thus $Z = \bigcap_1^\infty O_n$, $O_n \in (\tau W(\mathcal{L}))'$. Hence $O_n = \bigcup_\alpha W(L_{n_\alpha})', L_{n_\alpha} \in \mathcal{L}$. Thus $Z \in \bigcup_\alpha W(L_{n_\alpha})$. Z has a finite cover $\bigcup_1^N W(L_{n_{\alpha i}})' = W(L)$. Hence, $Z \subset W(L_n)' \subset O_n$ for all n and consequently, $Z \subset \bigcap_1^\infty W(L_n)' \subset \bigcap_1^\infty O_n = Z$. Thus $Z = \bigcap_1^\infty W(L_n)'$, i.e., Z is a countable intersection of $W(L_n)' L_n \in \mathcal{L}$, or $Z \in \sigma(W(\mathcal{L}))$.

Now, let us modify lightly the W mapping and consider $W(\sigma, A) = \{\mu \in IR(\sigma, \mathcal{L}); \mu(A) = 1\} \equiv W(A) \cap IR(\sigma, \mathcal{L})$. For $\mu \in MR(\mathcal{L})$ define μ' on $\mathfrak{A}(W(\sigma, \mathcal{L})) = W(\sigma, \mathfrak{A}(\mathcal{L}))$ by $\mu'(W(\sigma, B)) = \mu(B), B \in \mathfrak{A}(\mathcal{L})$.

PROPOSITION 3.7. Let $\mathcal{L}$ be a separating and disjunctive lattice. Then the following statements are equivalent:

1) $\mu' \in MR(W(\sigma,\mathcal{L}))$ for all $\mu \in MR(\mathcal{L})$
2) If $\rho \in MR(W(\sigma,\mathcal{L}))$ then $\rho = \mu'$, where $\mu \in MR(\mathcal{L})$
3) $\mu \in MR(\sigma,\mathcal{L})$ if and only if $\mu' \in MR(\sigma, W(\sigma,\mathcal{L}))$
4) If $\mu \in MR(\sigma,\mathcal{L})$, then μ' is the projection of $\hat{\mu}$ on $IR(\sigma,\mathcal{L})$ (since $\hat{\mu}^*(IR(\sigma,\mathcal{L})) = \hat{\mu}(IR(\mathcal{L}))$ in this case).

The proof of the equivalence is not difficult (see [5]).

4. ALEXANDROV'S REPRESENTATION THEOREM AND WEAK CONVERGENCE.

In this section we summarize some of the properties of weak convergence of measures due to Alexandrov and investigate the relationship of these properties to the induced measures on $IR(\mathcal{L})$ and $IR(\sigma,\mathcal{L})$ considered in the previous section, i.e., $\hat{\mu}$, $\tilde{\mu}$, and μ' respectively.

Let X be an abstract set and let $\mathcal{L}$ be a δ normal lattice. The algebra of all $\mathcal{L}$-continuous functions is denoted by $C(\mathcal{L})$; the algebra of bounded $\mathcal{L}$-continuous functions is denoted by $C_b(\mathcal{L})$.

We state for reference Alexandrov's Representation Theorem (A.R.T.).

THEOREM (Alexandrov) [1b]. Let $\mathcal{L}$ be a δ normal lattice. Then, the conjugate space $C_b(\mathcal{L})'$ of $C_b(\mathcal{L})$ is $MR(\mathcal{L})$. In more details: To every bounded linear functional Φ there corresponds a unique $\mu \in MR(\mathcal{L})$ such that $\Phi(f) = \int f d\mu$ with $\|\Phi\| = |\mu|$. The positive and negative parts of Φ correspond to those of μ. Furthermore, if Φ is non-negative, then $\forall A \in \mathcal{L}$, $\mu(A) = \inf \Phi(f)$ where $\inf$ is taken over all f in $C_b(\mathcal{L})$ such that $\chi_A \le f \le 1$, where χ_A is the characteristic function of A.

The spaces $C(\mathcal{L})$ and $C_b(\mathcal{L})$ are vector spaces. In particular, $C_b(\mathcal{L})$ is a Banach space with sup norm. We can topologize $MR(\mathcal{L})$ with the (weak *) topology as follows: If $\mu \in MR(\mathcal{L})$, then $\mu_\alpha \in MR(\mathcal{L})$ converges to μ in the weak topology if and only if $\int f d\mu_\alpha$ converges to $\int f d\mu$ for all $f \in C_b(\mathcal{L})$. In other words, we write $\mu_\alpha \in MR(\mathcal{L})$ and $\mu_\alpha \xrightarrow{w} \mu$: iff $\int f d\mu_\alpha \to \int f d\mu$ for all $f \in C_b(\mathcal{L})$.

PROPOSITION 4.1 (Portmanteau) [1c]. Let $\{\mu_\alpha\}$ be a net in $MR^+(\mathcal{L})$ the set of all non-negative measures of $MR(\mathcal{L})$. The following statements are equivalent:

1) $\mu_\alpha \xrightarrow{w} \mu_o$
2) $\mu_\alpha(X) \to \mu_o(X)$ and $\overline{\lim}\ \mu_\alpha(L) \le \mu_o(L)$ for all $L \in \mathcal{L}$
3) $\mu_\alpha(X) \to \mu_o(X)$ and $\underline{\lim}\ \mu_\alpha(L') \ge \mu_o(L')$ for all $L \in \mathcal{L}$

In what follows we assume that $\mathcal{L}$ is δ normal and disjunctive. Note the following facts:

1) $IR(\mathcal{L})$ is closed in $MR^+(\mathcal{L})$
2) If $\mathcal{L}$ is separating, then $[\overline{X}] = MR(\mathcal{L})$, where $[X]$ is the linear spacé spanned by all μ_x in $MR(\mathcal{L})$ and the closure is taken with respect to the weak topology.

These statements are not difficult to prove. In fact, with regards to 1), we have to use the W-compactness of $IR(\mathcal{L})$ and the following

PROPOSITION 4.2. Let $\mu_\alpha, \mu \in IR(\mathcal{L})$. Then,

$$\mu_\alpha \xrightarrow{W} \mu \text{ if and only if } \mu_\alpha \xrightarrow{w} \mu$$

PROOF. Let $\mu_\alpha \xrightarrow{W} \mu$ and let $L \in \mathcal{L}$ be such that $\mu(L') = 1$. Then, $\mu \in W(L)'$. Since $W(L)'$ is W-open, hence $\mu_\alpha \in W(L)'$ for all α, $\alpha \ge \alpha_o$, i.e., $\underline{\lim}\ \mu_\alpha(L') = 1 \ge \mu(L') = 1$. Hence $\underline{\lim}\ \mu_\alpha(L') \ge \mu(L'), \forall L \in \mathcal{L}$. Obviously, $\lim \mu_\alpha(X) = 1 = \mu(X)$. By Proposition 4.1, we have $\mu_\alpha \xrightarrow{w} \mu$.

Conversely, suppose $\mu_\alpha \xrightarrow{w} \mu$ and let $\mu \in W(L)'$. Then, $\mu(L') = 1$ and by Proposition 4.1, $\lim \mu_\alpha(L') = 1$. Thus, $\lim \mu_\alpha(L') = 1$ and $\mu_\alpha \in W(L)'$ for all $\alpha, \alpha \ge \alpha_o$. Hence $\mu_\alpha \xrightarrow{W} \mu$ because the open sets are generated by $W(L')$.

COROLLARY 4.1. Proposition 4.2 holds if the topological space $\{IR(\mathcal{L}), W\}$ is replaced by its subspace $\{IR(\sigma,\mathcal{L}), W\}$.

Next, we consider the situation with two δ normal lattices $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_1 \subset \mathcal{L}_2$.

Set $\mu^{\cdot}(E) = inf\{\mu(L'): E \subset L', L \in \mathcal{L}_1\}$. Similarly, define $\mu_{\cdot}(E) = sup\{\mu(L): L \subset E, L \in \mathcal{L}_1\}$ for $E \subset X$.

PROPOSITION 4.3. Let $\mathcal{L}_1$ semiseparate $\mathcal{L}_2$. If $B \in \mathcal{L}_2$, μ_α, $\mu \in MR^+(\mathcal{L}_1)$ and $\mu_\alpha \xrightarrow{w} \mu$, then $\mu^{\cdot}(B) \geq \overline{lim}\ \mu^{\cdot}_\alpha(B)$ and $\mu_{\cdot}(B') \leq \underline{lim}(\mu_\alpha)_{\cdot}$.

PROOF. Since $\mathcal{L}_1$ semiseparates $\mathcal{L}_2$, then for every $B \in \mathcal{L}_2$ and $L \in \mathcal{L}_1$, $B \subset \tilde{L} \subset L'$, $\tilde{L} \in \mathcal{L}_1$. Hence

$$\mu^{\cdot}(B) = inf\ \{\mu(L):\ B \subset \tilde{L},\ \tilde{L} \in \mathcal{L}_1\}.$$

Similarly, we get

$$\mu_{\cdot}(B) = sup\ \{\mu(A'):\ A' \subset B,\ A \in \mathcal{L}_1\}.$$

Now, the conclusion of the proposition follows from the Portmanteau theorem.

PROPOSITION 4.4. Let $\mathcal{L}_1$ separate $\mathcal{L}_2$, and for μ_α, $\mu \in MR^+(\mathcal{L}_1)$ let ν_α, $\nu \in MR(\mathcal{L}_2)$ denote the extensions of μ_α, μ to $\mathfrak{A}(\mathcal{L}_2)$. Then

$$\mu_\alpha \xrightarrow{w} \mu \Rightarrow \nu_\alpha \xrightarrow{w} \nu.$$

PROOF. By Theorem 2.1 ν_α and ν, respectively are determined uniquely. Since $\mathcal{L}_1$ semiseparates $\mathcal{L}_2$, we have

$$\mu^{\cdot}(E) = inf\{\mu(A'):\ E \subset A',\ A \in \mathcal{L}_1\} = inf\{\mu(\tilde{A}):\ E \subset \tilde{A},\ \tilde{A} \in \mathcal{L}_1\}.$$

However, $\nu = \mu^{\cdot}$ on $\mathcal{L}_2$, and $\nu = \mu_{\cdot}$ on $\mathcal{L}_2$. By Proposition 4.3 we have $\mu^{\cdot}(B) \geq \overline{lim}\mu^{\cdot}_\alpha(B)$ or $\nu(B) \geq \overline{lim}\nu_\alpha(B)$ for $B \in \mathcal{L}_2$.

On the other hand, $\mu_\alpha(X) \to \mu(X)$, $\mu_\alpha(X) = \nu_\alpha(X)$, $\mu(X) = \nu(X)$. By the Portmanteau theorem we have

$$\nu_\alpha \xrightarrow{w} \nu$$

It is of interest to note that if $\mu \in M(\mathcal{L})$, then $\Phi(f) = \int_X f d\mu$ for $f \in C_b(\mathcal{L})$ is a bounded linear functional on $C_b(\mathcal{L})$, and even a positive linear functional if $\mu \geq 0$. By A.R.T. $\Phi(f) = \int_X f d\nu$, where $\nu \in MR(\mathcal{L})$, and it is not difficult to see that $\mu \leq \nu$ on $\mathcal{L}$ and $\mu(X) = \nu(X)$.

PROPOSITION 4.5. If $\mathcal{L}$ is a strongly normal lattice of subsets of X and $\mu_\alpha \in MR(\mathcal{L})$, then

$$\mu_\alpha \xrightarrow{w} \mu \Leftrightarrow \tilde{\mu}_\alpha \xrightarrow{w} \tilde{\mu}.$$

PROOF. Let $f \in C_b(\mathcal{L})$. Define $\hat{f}$ on $IR(\mathcal{L})$ by $\hat{f}(\mu) = \int_X f d\mu$. Since $\hat{f}(\mu_x) = \int f d\mu_x = f(x)$, then $\hat{f}$ extends f. Also, $\hat{f}$ is continuous with respect to the weak topology, i.e., (be definition)

$$\mu_\alpha \xrightarrow{w} \mu \Rightarrow \hat{f}(\mu_\alpha) = \int f d\mu_\alpha \to \int f d\mu = \hat{f}(\mu).$$

Without loss of generality we can assume $f \geq 0$ and $\mu \geq 0$. We show that $\int f d\mu = \int \hat{f} d\tilde{\mu}$, for all $f \in C_b(\mathcal{L})$. Note that the set $\{\hat{f}: f \in C_b(\mathcal{L})\}$ with the sup norm is a subalgebra of the Banach algebra $C(\tau W(\mathcal{L}))$ and is isometrically isomorphic to the Banach algebra $C_b(\mathcal{L})$. Moreover, by the Stone-Weierstrass theorem $\{\hat{f}: f \in C_b(\mathcal{L})\}$ is a dense subset of $C(\tau W(\mathcal{L}))$. Thus, we have $C(\tau W(\mathcal{L})) = \{\hat{f}: f \in C_b(\mathcal{L})\}$.

Let Φ be a bounded linear functional on $C_b(\mathcal{L})$. Thus, by A.R.T., $\Phi(f) = \int f d\mu$, where $\mu \in MR(\mathcal{L})$. Define $\hat{\Phi}$ on $C(\tau W(\mathcal{L}))$ by

$$\hat{\Phi}(\hat{f}) = \Phi(f) \text{ for } f \in C_b(\mathcal{L}).$$

Clearly, $\hat{\Phi}$ is a bounded linear functional. Again, by A.R.T.

$$\hat{\Phi}(\hat{f}) = \int f d\tilde{\nu}$$

where $\tilde{\nu} \in MR(\tau W(\mathcal{L}))$ and $\tilde{\nu}(W(L)) = inf\{\Phi(f): \chi_{W(L)} \leq f \leq 1\}$

$$= inf\{\Phi(f): \chi_L \leq f \leq 1\} = \mu(L) = \tilde{\mu}(W(L)).$$

Hence, $\tilde{\nu} = \tilde{\mu}$ and $\hat{\Phi}(\hat{f}) = \Phi(f) = \int \hat{f} d\tilde{\mu} = \int f d\mu$.

Thus $\quad \mu_\alpha \xrightarrow{w} \mu \Leftrightarrow \tilde{\mu}_\alpha \xrightarrow{w} \tilde{\mu}$.

Proposition 4.4 and Proposition 4.5 together give an alternative proof of a theorem of Kirk and Crenshaw in the following formulation.

COROLLARY 4.5.1. Let $\mathcal{L}$ be a strongly normal lattice of subsets of X and $\tau W(\mathcal{L})$ be the Wallman topology on $IR(\mathcal{L})$. Let $\{\mu_\alpha\}$ be a net in $MR^+(\mathcal{L})$ and $\mu \in MR^+(\mathcal{L})$. The following are equivalent:

1) $\hat{\mu}_\alpha \xrightarrow{w} \hat{\mu}$
2) $\mu_\alpha(X) \to \mu(X)$ and $\overline{lim}\ \mu_\alpha(L) \leq \mu(L)$ for all $L \in \mathcal{L}$
3) $\mu_\alpha(X) \to \mu(X)$ and $\underline{lim}\ \mu_\alpha(L') \geq \mu(L')$ for all $L \in \mathcal{L}$

PROOF. Let μ_α, $\mu \in MR(\mathcal{L})$ and suppose that $\hat{\mu}_\alpha \xrightarrow{w} \hat{\mu}$ where we are, of course, referring here to the lattice $W(\mathcal{L})$ and the space $C_b(W(\mathcal{L}))$. Then, since $W(\mathcal{L})$ separates $\tau W(\mathcal{L})$, Proposition 4.4 gives $\tilde{\mu}_\alpha \xrightarrow{w} \tilde{\mu}$ which is equivalent to $\mu_\alpha \xrightarrow{w} \mu$ (by Proposition 4.5). This in conjunction with the Portmanteau theorem completes the proof.

We introduce a set of measures $M\widetilde{R}(\mathcal{L})$ as follows:

$M\widetilde{R}(\mathcal{L}) = \{\mu \in MR(\mathcal{L})$ and for any $\rho \in IR(\mathcal{L}) - IR(\sigma, \mathcal{L})$, there exists a $G \in (\tau W(\mathcal{L}))'$ such that $\rho \in G$ and $|\mu|(G) = 0\}$.

Note that the measures of $M\widetilde{R}(\mathcal{L})$ integrate all $f \in C(\mathcal{L})$.

Let $\mu_o \in M\widetilde{R}(\mathcal{L})$, and let $\mathcal{O}: \{V(\mu_o, f_1, f_2, \ldots, f_n, \varepsilon) = \mu \in MR(\mathcal{L}): |\int_X f_i d\mu - \int_X f_i d\mu_o| < \varepsilon$, where $f_i \in C(\mathcal{L}), (1 \leq i \leq n)\}$ be a neighborhood system at point μ_o. $\mathcal{O}$ is a base for topology $\widetilde{\mathcal{O}}$ on $M\widetilde{R}(\mathcal{L})$. Clearly, $\widetilde{\mathcal{O}}$ and $(\tau W(\sigma, \mathcal{L}))'$ coincide on $IR(\sigma, \mathcal{L})$.

PROPOSITION 4.6. Let $\mu_\alpha, \mu_o \in M\widetilde{R}(\mathcal{L})$. Then,

$$\mu_\alpha \xrightarrow{\widetilde{\mathcal{O}}} \mu_o \Leftrightarrow \mu_\alpha \xrightarrow{w} \mu_o$$

PROOF. By definition, a net $\{\mu_\alpha\}$ on $M\widetilde{R}(\mathcal{L})$ converges to $\mu \in M\widetilde{R}(\mathcal{L})$ with respect to $\mathcal{O}$ if and only if $\int_X f d\mu_\alpha \to \int_X f d\mu$ for all $f \in C(\mathcal{L})$. Therefore, if $\mu_\alpha \xrightarrow{\widetilde{\mathcal{O}}} \mu_o$, then clearly $\mu_\alpha \xrightarrow{w} \mu_o$.

Conversely, let $\mu_\alpha \xrightarrow{w} \mu_o$. We have to prove that $\mu_\alpha \xrightarrow{\widetilde{\mathcal{O}}} \mu_o$ or, equivalently, that the functional

$$\hat{f}(\mu) = \int_X f d\mu, \ \mu \in IR(\sigma, \mathcal{L})$$

is continuous with respect to $\tau W(\sigma, \mathcal{L})$, for all $f \in C(\mathcal{L})$.

First we show that $\hat{f}: IR(\sigma, \mathcal{L}) \to R$.

Let $F_n = \{x \in X: |f(x)| \leq n\} \in \mathcal{L}$. Since $f \in C(\mathcal{L})$, then $F_n \uparrow X$ and consequently, $\mu(F_n) \uparrow 1$, i.e., there exists N such that $\mu(F_n) = 1$ for $n \geq N$. Thus,

$$|\hat{f}(\mu)| = |\int_X f d\mu| = |\int_{F_n} f d\mu| \leq \int_{F_n} |f| d\mu \leq N.$$

Next, we show that $\hat{f}$ is continuous. Assume w.l.o.g. $f \geq 0$. Let $L_n = \{x: f(x) \geq n\}$ where $L_n \in \mathcal{L}$ and $L_n \downarrow \emptyset$. Then, $\mu_o(L_n) = 0$ for some N. Clearly, $\mu_\alpha(L_n) = 0$ for all $\alpha > \alpha_n$ and

$$|\hat{f}(\mu_\alpha) - \hat{f}(\mu_o)| = |\int_X f d\mu_\alpha - \int_X f d\mu_o| = |\int_X f_N d\mu_\alpha - \int_X f_N d\mu_o|$$

for all $\alpha > \alpha_n$, where $f_N = f \wedge N \in C_b(\mathcal{L})$. Since $\mu_\alpha \xrightarrow{w} \mu_o$, we have $|f(\mu_\alpha) - f(\mu_o)| \to 0$. Thus $\mu_\alpha \xrightarrow{\widetilde{\mathfrak{S}}} \mu_o$.

Let X and Y be topological spaces and $\mathcal{L}_1$ and $\mathcal{L}_2$ be lattices of closed subsets of X and Y respectively. Suppose T is a linear mapping of $M\widetilde{R}(\mathcal{L}_1)$ onto $M\widetilde{R}(\mathcal{L}_2)$ and 1:1 such that $\|T\mu\| = |\mu|$ and T is continuous both ways in the respective topologies, i.e,. T is $\widetilde{\mathfrak{S}}_1 - \widetilde{\mathfrak{S}}_2$ homeomorphic. $\widetilde{\mathfrak{S}}_i$ is a neighborhood system at point $\mu_i, i = 1,2$ which forms a basis for the topology of $M\widetilde{R}(\mathcal{L}_i)$. By Proposition 4.6 $\widetilde{\mathfrak{S}}_i$ topology restricted to $IR(\sigma, \mathcal{L}_i)$ yields the same closed sets as $\tau W(\sigma, \mathcal{L}_i)$. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are separating, disjunctive and replete, then we can identify $IR(\sigma, \mathcal{L}_1) = X$ and $IR(\sigma, \mathcal{L}_2) = Y$.

PROPOSITION 4.7. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be separating, disjunctive and replete. If $M\widetilde{R}(\mathcal{L}_1)$ and $M\widetilde{R}(\mathcal{L}_2)$ are isomorphic, then X and Y are homeomorphic with respect to the $\tau\mathcal{L}_1$ and $\tau\mathcal{L}_2$ topologies of closed sets.

The proof follows immediately from Proposition 4.6 and the definitions of the relevant topologies.

This isomorphism proposition gives the following results:

1) If X is $T_{3\frac{1}{2}}$ (a Tychonov space) and $\mathcal{L}_1 = \mathcal{Z}_1$ and if Y is $T_{3\frac{1}{2}}$ and $\mathcal{L}_2 = \mathcal{Z}_2$ where $\mathcal{L}_1$ and $\mathcal{L}_2$ are replete, i.e., X and Y are real-compact, then $M\widetilde{R}(\mathcal{Z}_1)$ and $M\widetilde{R}(\mathcal{Z}_2)$ isomorphic implies that X and Y are homeomorphic [9].

2) If X, $\mathcal{L}_1(=\mathcal{F}_1)$ and Y, $\mathcal{L}_2(=\mathcal{F}_2)$ are T_4 spaces and each $\mathcal{L}_i$ is replete, then $M\widetilde{R}(\mathcal{F}_1)$ and $M\widetilde{R}(\mathcal{F}_2)$ isomorphic implies that X and Y are homeomorphic.

We now turn attention to $C(\mathcal{L})$; unlike the situation with $C_b(\mathcal{L})$ we have

$$C(\mathcal{L}) \leftrightarrow C(W(\sigma, \mathcal{L}))$$

$$f \leftrightarrow \hat{f}$$

where $\hat{f}(\mu) = \int_X f d\mu, \mu \in IR(\sigma, \mathcal{L})$, i.e., $C(\mathcal{L})$ is algebraically isomorphic to $C(W(\sigma, \mathcal{L}))$. Details can be found in [3].

PROPOSITION 4.8. $\mu_\alpha \xrightarrow{w} \mu \Leftrightarrow \mu'_\alpha \xrightarrow{w} \mu'$ where $\mu'_\alpha, \mu \in MR(\sigma, \mathcal{L})$ for all $\mu \geq 0$.

PROOF. We have

$$\mu_\alpha \xrightarrow{w} \mu \Rightarrow \int_X f d\mu_\alpha \to \int_X f d\mu, \text{ for all } f \in C_b(\mathcal{L}).$$

On the other hand,

$$\mu'_\alpha \xrightarrow{w} \mu' \Leftrightarrow \int_{IR(\sigma,\mathcal{L})} \hat{f} d\mu'_\alpha \to \int_{IR(\sigma,\mathcal{L})} \hat{f} d\mu', \ f \in C_b(W(\sigma, \mathcal{L}),$$

since $C_b(\mathcal{L})$ and $C_b(W(\sigma, \mathcal{L})$ are isomorphic ($f \leftrightarrow \hat{f}$). Let $\Phi(f) = \int f d\mu, \mu \geq 0$. By A.R.T. define $\bar{\Phi}(\hat{f}) = \Phi(f)$ on $W(\sigma, \mathcal{L})$. Clearly, Φ is a bounded linear functional. Again, by A.R.T. $\bar{\Phi}(\hat{f}) = \int_{IR(\sigma,\mathcal{L})} f d\rho$ where $\rho \in MR(W(\sigma, \mathcal{L}))$ and $\rho(W(L)) = inf \bar{\Phi}(\hat{f}) = inf \Phi(f) = \mu(L) = \mu'(W(\sigma, L))$ $\chi_{W(\sigma,\mathcal{L})} \leq \hat{f} \leq 1, \chi_L \leq f \leq 1$. Thus $\bar{\Phi}(\hat{f}) = \Phi(f) = \int \hat{f} d\mu' = \int f d\mu$. Hence,

$$\mu_\alpha \xrightarrow{w} \mu \Leftrightarrow \mu'_\alpha \xrightarrow{w} \mu'$$

REMARK. If $h \in C(\tau W(\sigma, \mathcal{L}))$, then $f = h|_X \in C(\tau\mathcal{L})$, and if $f \in C(\mathcal{L})$ then $h = \hat{f}$ and $h \in C(W(\sigma, \mathcal{L}))$. This situation arises, for example, if X is $T_{3\frac{1}{2}}$ space and $\mathcal{L} = \mathcal{Z}$ (the lattice of zero sets of X).

5. ALEXANDROV LATTICES.

Consider $X, \mathcal{L}$ where $\mathcal{L}$ is δ normal and complement generated (completely normal). Then Alexandrov's Fundamental Theorem [1b], states that $MR(\sigma,\mathcal{L})$ is weakly sequentially closed in $MR(\mathcal{L})$, i.e., if $\mu_n \in MR(\sigma,\mathcal{L})$ and $\mu_n \xrightarrow{w} \mu$, then $\mu \in MR(\sigma,\mathcal{L})$. We will call lattices for which this is true, Alexandrov lattices and will initiate a consideration of such lattices in this section. Formally, then we have

DEFINITION. A δ normal lattice $\mathcal{L}$ of subsets of X is said to be an Alexandrov lattice if $\mu_n \in MR(\sigma,\mathcal{L})$ and $\mu_n \xrightarrow{w} \mu$, where $\mu \in MR(\mathcal{L})$, imply $\mu \in MR(\sigma,\mathcal{L})$

PROPOSITION 5.1. Let $\mathcal{L}_1$ and $\mathcal{L}_2$, $\mathcal{L}_1 \subset \mathcal{L}_1$, be δ normal lattices. If $\mathcal{L}_2$ is $\mathcal{L}_1$ c.p. or c.b. and $\mathcal{L}_1$ s.s. $\mathcal{L}_2$, then if $\mathcal{L}_1$ is an Alexandrov lattice, $\mathcal{L}_2$ is also an Alexandrov lattice.

PROOF. Suppose $\nu_n \in MR(\sigma,\mathcal{L}_2)$ and $\nu_n \xrightarrow{w} \nu$, $\nu \in MR(\mathcal{L}_2)$. Since $\mathcal{L}_1$ s.s. $\mathcal{L}_2$ and $C_b(\mathcal{L}_2) \supset C_b(\mathcal{L}_1)$, then we have

$$\nu_n \xrightarrow{w} \mu,\ \mu_n = \nu_n\big| \in MR(\sigma,\mathcal{L}_1),\ \mu = \nu\,|\,.$$

Since $\mathcal{L}_1$ is Alexandrov, $\mu \in MR(\sigma,\mathcal{L}_1)$ and consequently, $\nu \in MR(\sigma,\mathcal{L}_2)$ (since $\mathcal{L}_2$ is $\mathcal{L}_1$ c.p. or c.b.). Thus $\mathcal{L}_2$ is also an Alexandrov lattice.

REMARK. If instead of $\mathcal{L}_1$ s.s. $\mathcal{L}_2$ we assume that $\mathcal{L}_1$ is δ and $\sigma(\mathcal{L}_1) \subset s(\mathcal{L}_1)$, then in this case $\mu_n = \nu_n \mid \in M(\sigma,\mathcal{L}_1)$ and therefore, by Choquet's capacity theorem [7] $\mu_n \in MR(\sigma,\mathcal{L}_1)$. Also, $\mu_n \xrightarrow{w} \mu = \nu \mid \in M(\mathcal{L}_1)$, but $\mu \le \rho$ on $\mathcal{L}_1$, where $\rho \in MR(\mathcal{L}_1)$ and $\mu(X) = \rho(X)$, and since $\int f d\mu = \int f d\rho$ for all $f \in C_b(\mathcal{L}_1)$, $\mu_n \xrightarrow{w} \rho$. Hence $\rho \in MR(\sigma,\mathcal{L}_1)$ since $\mathcal{L}_1$ is Alexandrov and consequently $\mu \in M(\sigma,\mathcal{L}_1)$.

Note that if $\mathcal{L}_1$ and $\mathcal{L}_2$ are δ normal and $C(\mathcal{L}_1) = C(\mathcal{L}_2)$, which implies that $\mathcal{L}_1$ separates $\mathcal{L}_2$, and if $\mathcal{L}_2$ is c.p., then $\mathcal{L}_2$ is $\mathcal{L}_1$ c.p. and $\mu \in MR(\sigma,\mathcal{L}_1)$. Then by Theorem 2.1 μ extends uniquely to $\nu \in MR(\sigma,\mathcal{L}_2)$. In other words, we have

COROLLARY 5.1. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be δ normal, $C(\mathcal{L}_2) = C(\mathcal{L}_1)$ and $\mathcal{L}_2$ be c.p.. Then,

$$\mathcal{L}_1 \textit{ Alexandrov} \Rightarrow \mathcal{L}_2 \textit{ Alexandrov.}$$

By Proposition 5.1 we also have the following

COROLLARY 5.2. If $\mathcal{L}$ is δ normal and c.p. then $\mathcal{L}$ is $\mathfrak{Z}(\mathcal{L})$ c.p. and $\mathcal{L}$ is also Alexandrov since $\mathfrak{Z}(\mathcal{L})$ is Alexandrov.

Suppose $\mathcal{L}_1$ and $\mathcal{L}_2$ are δ normal lattices, $\mathcal{L}_1 \subset \mathcal{L}_2$ and $C(\mathcal{L}_1) = C(\mathcal{L}_2)$. Let $\mu_n \xrightarrow{w} \mu$ where $\mu_n \in MR(\sigma,\mathcal{L}_1)$ and $\mu \in MR(\mathcal{L}_1)$. Then, if $\nu_n \in MR(\sigma,\mathcal{L}_2)$ is the unique extension of μ_n, and ν that of μ, we have $\nu_n \xrightarrow{w} \nu$ and $\nu \in MR(\sigma,\mathcal{L}_2)$ assuming $\mathcal{L}_2$ is Alexandrov. Therefore, $\mu \in MR(\sigma,\mathcal{L}_1)$ and $\mathcal{L}_1$ is Alexandrov. This fact together with Corollary 5.1 gives

COROLLARY 5.3. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are δ normal lattices, $\mathcal{L}_2$ is c.p. and $C(\mathcal{L}_2) = C(\mathcal{L}_1)$, then $\mathcal{L}_1$ is Alexandrov if and only if $\mathcal{L}_2$ is Alexandrov.

PROPOSITION 5.2. If $\mathcal{L}$ is a δ normal and c.p. lattice of subsets of X and $\mu_n \xrightarrow{w} \mu$ where $\mu_n \in M(\sigma,\mathcal{L}), \mu \in M(\mathcal{L})$, then $\mu \in M(\sigma,\mathcal{L})$.

PROOF. Let $\mu_n \in M(\sigma,\mathcal{L})$ and $\mu_n \xrightarrow{w} \mu$. The functional $\phi_n(f) = \int f d\mu_n$ is a bounded linear functional of $f \in C_b(\mathcal{L})$, and by A.R.T., we have $\Phi_n(f) = \int f d\mu_n = \int f d\nu_n$, $\nu_n \in MR(\mathcal{L})$, $\mu_n(X) = \nu_n(X)$ and $\mu_n \le \nu_n$ on $\mathcal{L}$. Since $\mathcal{L}$ is c.p., we also have $\nu_n \in MR(\sigma,\mathcal{L})$. Thus $\int f d\nu_n = \int f d\mu_n \to \int f d\mu$. Also, by A.R.T., $\Phi(f) = \int f d\mu = \int f d\nu, \nu \in MR(\mathcal{L})$ and $\mu(X) = \nu(X)$ on $\mathcal{L}$. Therefore, $\int f d\nu_n \to \int f d\nu$ or $\nu_n \xrightarrow{w} \nu$. Since $\mathcal{L}$ is δ normal and c.p., by Alexandrov's theorem $\nu \in MR(\sigma,\mathcal{L})$. On the other hand, we have $\mu \le \nu$ on $\mathcal{L}$. Therefore, $\mu \in M(\sigma,\mathcal{L})$.

PROPOSITION 5.3. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be lattices of subsets of X. Suppose $\mathcal{L}_1$ separates $\mathcal{L}_2$ and $\mathcal{L}_1$ is an Alexandrov lattice. If $M(\sigma,\mathcal{L}'_2) \cap MR(\mathcal{L}_2) \subset M(\sigma,\mathcal{L}_2)$, then $\mathcal{L}_2$ is also an Alexandrov lattice.

PROOF. Let $\nu_n \xrightarrow{w} \nu$ where $\nu_n \in MR(\sigma,\mathcal{L}_2)$ and $\nu \in MR(\mathcal{L}_2)$. Since $\mathcal{L}_1$ is an Alexandrov lattice and $\nu_n| = \mu_n \in MR(\sigma,\mathcal{L}_1)$, we have $\nu_n| = \mu_n \xrightarrow{w} \nu| = \mu \in MR(\sigma,\mathcal{L}_1)$. $\mathcal{L}_1$ actually coseparates $\mathcal{L}_2$ since $\mathcal{L}_1$ separates $\mathcal{L}_2$ and $\mathcal{L}_1$ is normal. It is not difficult to see that $\nu \in M(\sigma,\mathcal{L}'_2)$ since ν must be $\mathcal{L}_1$-regular on $\mathcal{L}'_2$. Now, since $\nu \in MR(\mathcal{L}_2)$ and $\nu \in M(\sigma,\mathcal{L}'_2)$, we have $\nu \in M(\sigma,\mathcal{L}'_2) \cap MR(\mathcal{L}_2)$. Clearly, if $M(\sigma,\mathcal{L}'_2) \cap MR(\mathcal{L}_2) \subset M(\sigma,\mathcal{L}_2)$, then $\nu \in MR(\sigma,\mathcal{L}_2)$. Therefore, $\mathcal{L}_2$ is an Alexandrov lattice.

Let $\mathcal{L}_1$ be a lattice of subsets of X and $\mathcal{L}_2$ be a lattice of subsets of Y. Again, we assume that $\mathcal{L}_1$ and $\mathcal{L}_2$ are δ normal.

Let $T: X \to Y$ be $\mathcal{L}_1 - \mathcal{L}_2$ continuous. Consider a mapping $A: C_b(\mathcal{L}_2) \to C_b(\mathcal{L}_1)$, such that A is linear and bounded.

If the mapping A is defined by $Ag = gT$ where $g \in C_b(\mathcal{L}_2)$, then define the adjoint map by $A': C_b(\mathcal{L}_1)' \to C_b(\mathcal{L}_2)'$ where $C_b(\mathcal{L}_i)'$ is congruent to $MR(\mathcal{L}_i)$ $(i = 1,2)$ and $(A'\Phi)(g) = \Phi(Ag)$. By A.R.T., we have

$$\Phi \leftrightarrow \mu,\ \mu \in MR(\mathcal{L}_1) \text{ and } A'\Phi \leftrightarrow \nu, \nu \in MR(\mathcal{L}_2). \text{ Then,}$$

$$\Phi(AG) = \int Agd\mu \text{ and } (A'\Phi)(g) = \int gd\nu, \text{ for all } g \in C_b(\mathcal{L}_2)$$

and consequently $A': MR(\mathcal{L}_1) \to MR(\mathcal{L}_2)$ where $A'\mu = \nu$ and

$$\int_Y gd\nu = (A'\Phi)(g) = \Phi(Ag) = \int_X Agd\mu = \int_X gTd\mu = \int_Y gd\mu T^{-1},\ g \in C_b(\mathcal{L}_2).$$

Note that A is a linear mapping and that $Ag_1g_2 = g_1Tg_2T = Ag_1Ag_2$. Therefore, A is an algebra homeomorphism. Also, we have $\|Ag\| = \|gT\| \leq \|g\|$. Indeed, A is bounded. If T s surjective, then $\|Ag\| = \|g\|$, i.e., A is an isometry, and consequently A is invertible.

Some basic properties of A' are collected in the following

PROPOSITION 5.4. a) If $\mu \geq 0$, then $\nu = A'\mu \geq 0$

b) $A'\mu = \nu \geq \mu T^{-1}$ on $\mathcal{L}_2$ and $\nu(Y) = \mu T^{-1}(Y)$

c) $A'(IR(\mathcal{L}_1)) \subset IR(\mathcal{L}_2)$

d) $A'|_{IR(\mathcal{L}_1)}$ is Wallman continuous.

PROOF. We show only b). Further details can be found in [2]. We have $\mu T^{-1}(L) = \int_L d\mu T^{-1} = \int_Y \chi_L d\mu T^{-1} \leq \int_Y gd\mu T^{-1} = \int_Y gd\nu$ where $g \in C_b(\mathcal{L}_2)$ and $\chi_L \leq g \leq 1$. Therefore, $\mu T^{-1}(L) \leq \nu(L)$ for all $L \in \mathcal{L}_2$. If $g = 1$, we obtain $\int_Y d\nu = \int_Y d\mu T^{-1}$. Hence $\nu(Y) = \mu T^{-1}(Y)$.

PROPOSITION 5.5. a) If $\mathcal{L}_2$ is c.p., then $A'(MR(\sigma,\mathcal{L}_1)) \subset MR(\sigma,\mathcal{L}_2)$

b) If T is surjective and $\mathcal{L}_2$ is $T^{-1}(\mathcal{L}_2)$ c.b., then $MR(\sigma,\mathcal{L}_2) \subset A'(MR(\sigma,\mathcal{L}_1))'$

c) If a) and b) hold, then $A'(MR(\sigma,\mathcal{L}_1)) = MR(\sigma,\mathcal{L}_2)$

PROOF. Here we show only a). Suppose $\mathcal{L}_2$ is c.p.. Let $\mu \in MR(\sigma,\mathcal{L}_1)$ and consider any element of $A'(MR(\sigma,\mathcal{L}_1))$, $A'\mu$. We must show that $A'\mu = \nu \in MR(\sigma,\mathcal{L}_2)$. By A.R.T., we have $\mu \leftrightarrow \Phi$ and Φ is σ-smooth. In fact, consider $\{g_n\}$, $g_n \in C_b(\mathcal{L}_2)$, $g_n \downarrow 0$. Then, $g_nT \downarrow 0$ and, therefore, $\lim \int g_nTd\mu = 0$. However, $\lim \int g_nTd\mu = \lim \int g_nd\nu = 0$ which means $\Phi \leftrightarrow \nu$ where $\Phi(g) = \int gd\nu$, for all $g \in C_b(\mathcal{L}_2)$. Since Φ is σ-smooth and $\mathcal{L}_2$ is c.p., we have $A'\mu = \nu \in MR(\sigma,\mathcal{L}_2)$. Hence

$$A'(MR(\sigma,\mathcal{L}_1)) \subset MR(\sigma,\mathcal{L}_2).$$

PROPOSITION 5.6. 1) Under the assumption a) of Proposition 5.5, if $\mathcal{L}_1$ is an Alexandrov lattice and

$$\mu_n \xrightarrow{w} \mu,\ \mu_n \in MR(\sigma,\mathcal{L}_1),$$

then $A'\mu_n \xrightarrow{w} A'\mu$ and $A'\mu_n$, $A'\mu \in MR(\sigma,\mathcal{L}_2)$;

2) Under the assumptions a) and b) of Proposition 5.5 and if A is surjective, then $\mathcal{L}_1$ Alexandrov implies that $\mathcal{L}_2$ is Alexandrov.

PROOF. 1) Since $\mathcal{L}_1$ is Alexandrov, we have

$$\mu_n \xrightarrow{w} \mu, \ \mu \in MR(\sigma, \mathcal{L}_1).$$

Then by Proposition 5.4 d) $A'\mu_n \to A'\mu$. By Proposition 5.5 a)

$$A'\mu_n \in MR(\sigma, \mathcal{L}_2) \text{ and } A'\mu \in A'(MR(\sigma, \mathcal{L}_1)) \subset MR(\sigma, \mathcal{L}_2).$$

2) Let $\nu_n \in MR(\sigma, \mathcal{L}_2)$. Then $\nu_n \xrightarrow{w} \nu \in MR(\mathcal{L}_2)$.
By Proposition 5.5 b) we have

$$\nu_n = A'\mu_n, \ \ \mu_n \in MR(\sigma, \mathcal{L}_1); \ \ \nu = A'\mu, \ \ \mu \in MR(\mathcal{L}_1).$$

Since A is surjective, we have

$$\int Agd\mu_n = \int gd\nu_n \to \int gd\nu = \int Agd\mu.$$

Hence

$$\mu_n \xrightarrow{w} \mu.$$

Since $\mathcal{L}_1$ is Alexandrov, $\mu \in MR(\sigma, \mathcal{L}_1)$. Therefore, by Proposition 5.5 a) $A'\mu = \nu \in MR(\sigma, \mathcal{L}_2)$. Hence $\mathcal{L}_2$ is Alexandrov.

Thus, under the above assumptions the measure defined on Alexandrov lattices is invariant under adjoint mappings.

REFERENCES

1. ALEXANDROV, A.D., Additive set functions in abstract spaces, Mat. Sb. (a)8, 50, (1940), 307-348; (b)9, 51, (1941), 563-628; (c)13, 55, (1943), 139-268.
2. BACHMAN, G. & STRATIGOS, P., Some applications of the adjoint to lattice regular measures, Annali di Matematica Pura ed Applicata IV, Vol. 138 (1984), 379-397.
3. BACHMAN, G. & SULTAN, A., Extensions of regular lattice measures with topological applications, J. Math. Anal. Appl. 57 (1977), 539-559.
4. BACHMAN, G. & SULTAN, A., On regular extensions of measures, Pacific J. Math. 2 (1980), 389-395.
5. BACHMAN, G. & SZETO, M., On strongly measure replete lattices and the general Wallman remainder, Fundamenta Mathematicae, CXXII (1984), 199-217.
6. BERGSTROM, H., Weak Convergence of Measures, Academic Press, New York, London, 1982.
7. CHOQUET, G., Theory of capacities, Ann. Inst. Fourier 5 (1953-1954), 131-295.
8. KIRK, R.B. & CRENSHAW, J.A., A generalized topological measure theory, Trans. Amer. Math. Soc. Vol. 207 (1975), 189-217.
9. VARADARAJAN, V.S., Measures on topological spaces, Amer. Math. Soc. Transl. (2) 48, (1965), 161-228.